

# Second order parameter-uniform convergence for a finite difference method for a singularly perturbed linear parabolic system

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## Abstract A

singularly perturbed linear system of second order partial differential equations of parabolic reaction-diffusion type with given initial and boundary conditions is considered. The leading term of each equation is multiplied by a small positive parameter. These singular perturbation parameters are assumed to be distinct. The components of the solution exhibit overlapping layers. Shishkin piecewise-uniform meshes are introduced, which are used in conjunction with a classical finite difference discretisation, to construct a numerical method for solving this problem. It is proved that the numerical approximations obtained with this method are first order convergent in time and essentially second order convergent in the space variable uniformly with respect to all of the parameters.

## 1 Introduction

The following parabolic initial-boundary value problem is considered for a singularly perturbed linear system of second order differential equations

$$\frac{\partial \mathbf{u}}{\partial t} - E \frac{\partial^2 \mathbf{u}}{\partial x^2} + A\mathbf{u} = \mathbf{f}, \text{ on } \Omega, \quad \mathbf{u} \text{ given on } \Gamma, \quad (1)$$

where  $\Omega = \{(x, t) : 0 < x < 1, 0 < t \leq T\}$ ,  $\overline{\Omega} = \Omega \cup \Gamma$ ,  $\Gamma = \Gamma_L \cup \Gamma_B \cup \Gamma_R$  with  $\mathbf{u}(0, t) = \phi_L(t)$  on  $\Gamma_L = \{(0, t) : 0 \leq t \leq T\}$ ,  $\mathbf{u}(x, 0) = \phi_B(x)$  on  $\Gamma_B = \{(x, 0) : 0 \leq x \leq 1\}$ ,  $\mathbf{u}(1, t) = \phi_R(t)$  on  $\Gamma_R = \{(1, t) : 0 \leq t \leq T\}$ . Here, for

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all  $(x, t) \in \overline{\Omega}$ ,  $\mathbf{u}(x, t)$  and  $\mathbf{f}(x, t)$  are column  $n$ -vectors,  $E$  and  $A(x, t)$  are  $n \times n$  matrices,  $E = \text{diag}(\boldsymbol{\varepsilon})$ ,  $\boldsymbol{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_n)$  with  $0 < \varepsilon_i \leq 1$  for all  $i = 1, \dots, n$ . The  $\varepsilon_i$  are assumed to be distinct and, for convenience, to have the ordering

$$\varepsilon_1 < \dots < \varepsilon_n.$$

Cases with some of the parameters coincident are not considered here. The problem can also be written in the operator form

$$\mathbf{L}\mathbf{u} = \mathbf{f} \text{ on } \Omega, \mathbf{u} \text{ given on } \Gamma,$$

where the operator  $\mathbf{L}$  is defined by

$$\mathbf{L} = \frac{\partial}{\partial t} - E \frac{\partial^2}{\partial x^2} + A.$$

For all  $(x, t) \in \overline{\Omega}$  it is assumed that the components  $a_{ij}(x, t)$  of  $A(x, t)$  satisfy the inequalities

$$a_{ii}(x, t) > \sum_{\substack{j \neq i \\ j=1}}^n |a_{ij}(x, t)| \text{ for } 1 \leq i \leq n, \text{ and } a_{ij}(x, t) \leq 0 \text{ for } i \neq j \quad (2)$$

and, for some  $\alpha$ ,

$$0 < \alpha < \min_{\substack{(x,t) \in \overline{\Omega} \\ 1 \leq i \leq n}} \left( \sum_{j=1}^n a_{ij}(x, t) \right). \quad (3)$$

It is also assumed, without loss of generality, that

$$\max_{1 \leq i \leq n} \sqrt{\varepsilon_i} \leq \frac{\sqrt{\alpha}}{6}. \quad (4)$$

The reduced problem corresponding to (1) is defined by

$$\frac{\partial \mathbf{u}_0}{\partial t} + A\mathbf{u}_0 = \mathbf{f}, \text{ on } \Omega, \mathbf{u}_0 = \mathbf{u} \text{ on } \Gamma_B. \quad (5)$$

The norms  $\|\mathbf{V}\| = \max_{1 \leq k \leq n} |V_k|$  for any  $n$ -vector  $\mathbf{V}$ ,  $\|y\|_D = \sup\{|y(x, t)| : (x, t) \in D\}$  for any scalar-valued function  $y$  and domain  $D$ , and  $\|\mathbf{y}\| = \max_{1 \leq k \leq n} \|y_k\|$  for any vector-valued function  $\mathbf{y}$  are introduced. When  $D = \overline{\Omega}$  or  $\Omega$  the subscript  $D$  is usually dropped. Throughout the paper  $C$  denotes a generic positive constant, which is independent of  $x, t$  and of all singular perturbation and discretization parameters. Furthermore, inequalities between vectors are understood in the componentwise sense. Whenever necessary the required smoothness of the problem data is assumed.

For a general introduction to parameter-uniform numerical methods for singular perturbation problems, see [1], [2] and [3]. The piecewise-uniform Shishkin meshes  $\Omega^{M,N}$  in the present paper have the elegant property that they reduce to uniform meshes when the parameters are not small. The problem posed in the present paper is also considered in [5], where parameter uniform convergence is proved, which is first order in time and essentially first order in space. The meshes used there do not have the above typical property of Shishkin meshes. The main result of the present paper is well known in the scalar case, when  $n = 1$ . It is established in [4] for the case  $n = 2$ . The proof below of first order convergence in the time variable and essentially second order convergence in the space variable, for general  $n$ , draws heavily on the analogous result in [6] for a reaction-diffusion system.

The plan of the paper is as follows. In the next two sections both standard and novel bounds on the smooth and singular components of the exact solution are obtained. The sharp estimates for the singular component in Lemma 7 are proved by mathematical induction, while interesting orderings of the points  $x_{i,j}$  are established in Lemma 5. In Section 4 piecewise-uniform Shishkin meshes are introduced, in Section 5 the discrete problem is defined and the discrete maximum principle, the discrete stability properties and a comparison result are established. In Section 6 an expression for the local truncation error is derived and standard estimates are stated. In Section 7 parameter-uniform estimates for the local truncation error of the smooth and singular components are obtained in a sequence of theorems. The section culminates with the statement and proof of the essentially second order parameter-uniform error estimate.

## 2 Standard analytical results

The operator  $\mathbf{L}$  satisfies the following maximum principle

**Lemma 1.** *Let  $A(x, t)$  satisfy (2) and (3). Let  $\psi$  be any function in the domain of  $\mathbf{L}$  such that  $\psi \geq 0$  on  $\Gamma$ . Then  $\mathbf{L}\psi(x) \geq 0$  on  $\Omega$  implies that  $\psi(x) \geq 0$  on  $\overline{\Omega}$ .*

*Proof.* Let  $i^*, x^*, t^*$  be such that  $\psi_{i^*}(x^*, t^*) = \min_i \min_{\overline{\Omega}} \psi_i(x, t)$  and assume that the lemma is false. Then  $\psi_{i^*}(x^*, t^*) < 0$ . From the hypotheses we have  $(x^*, t^*) \notin \Gamma$  and  $\frac{\partial^2 \psi_{i^*}}{\partial x^2}(x^*, t^*) \geq 0$ . Thus

$$(\mathbf{L}\psi(x^*, t^*))_{i^*} = \frac{\partial \psi_{i^*}}{\partial t}(x^*, t^*) - \varepsilon_{i^*} \frac{\partial^2 \psi_{i^*}}{\partial x^2}(x^*, t^*) + \sum_{j=1}^n a_{i^*,j}(x^*, t^*) \psi_j(x^*, t^*) < 0,$$

which contradicts the assumption and proves the result for  $\mathbf{L}$ . ■

Let  $\tilde{A}(x, t)$  be any principal sub-matrix of  $A(x, t)$  and  $\tilde{\mathbf{L}}$  the corresponding operator. To see that any  $\tilde{\mathbf{L}}$  satisfies the same maximum principle as  $\mathbf{L}$ , it suffices to observe that the elements of  $\tilde{A}(x, t)$  satisfy *a fortiori* the same inequalities as those of  $A(x, t)$ .

**Lemma 2.** *Let  $A(x, t)$  satisfy (2) and (3). If  $\psi$  is any function in the domain of  $\mathbf{L}$ , then, for each  $i$ ,  $1 \leq i \leq n$  and  $(x, t) \in \overline{\Omega}$ ,*

$$|\psi_i(x, t)| \leq \max \left\{ \|\psi\|_T, \frac{1}{\alpha} \|\mathbf{L}\psi\| \right\}.$$

*Proof.* Define the two functions

$$\theta^\pm(x, t) = \max \left\{ \|\psi\|_T, \frac{1}{\alpha} \|\mathbf{L}\psi\| \right\} \mathbf{e} \pm \psi(x, t)$$

where  $\mathbf{e} = (1, \dots, 1)^T$  is the unit column vector. Using the properties of  $A$  it is not hard to verify that  $\theta^\pm \geq \mathbf{0}$  on  $\Gamma$  and  $\mathbf{L}\theta^\pm \geq \mathbf{0}$  on  $\Omega$ . It follows from Lemma 1 that  $\theta^\pm \geq \mathbf{0}$  on  $\overline{\Omega}$  as required. ■

A standard estimate of the exact solution and its derivatives is contained in the following lemma.

**Lemma 3.** *Let  $A(x, t)$  satisfy (2) and (3) and let  $\mathbf{u}$  be the exact solution of (1). Then, for all  $(x, t) \in \overline{\Omega}$  and each  $i = 1, \dots, n$ ,*

$$|\frac{\partial^l u_i}{\partial t^l}(x, t)| \leq C(\|\mathbf{u}\|_T + \sum_{q=0}^l \|\frac{\partial^q \mathbf{f}}{\partial t^q}\|), \quad l = 0, 1, 2$$

$$|\frac{\partial^l u_i}{\partial x^l}(x, t)| \leq C\varepsilon_i^{-\frac{l}{2}}(\|\mathbf{u}\|_T + \|\mathbf{f}\| + \|\frac{\partial \mathbf{f}}{\partial t}\|), \quad l = 1, 2$$

$$|\frac{\partial^l u_i}{\partial x^l}(x, t)| \leq C\varepsilon_i^{-1}\varepsilon_1^{-\frac{-(l-2)}{2}}(\|\mathbf{u}\|_T + \|\mathbf{f}\| + \|\frac{\partial \mathbf{f}}{\partial t}\| + \|\frac{\partial^2 \mathbf{f}}{\partial t^2}\| + \varepsilon_1^{\frac{l-2}{2}}\|\frac{\partial^{l-2} \mathbf{f}}{\partial x^{l-2}}\|), \quad l = 3, 4$$

$$|\frac{\partial^l u_i}{\partial t^{l-1} \partial x}(x, t)| \leq C\varepsilon_i^{\frac{1-l}{2}}(\|\mathbf{u}\|_T + \|\mathbf{f}\| + \|\frac{\partial \mathbf{f}}{\partial t}\| + \|\frac{\partial^2 \mathbf{f}}{\partial t^2}\|), \quad l = 2, 3.$$

*Proof.* The bound on  $\mathbf{u}$  is an immediate consequence of Lemma 2. Differentiating (1) partially with respect to 't' once and twice, and applying Lemma 2 the bounds  $\frac{\partial \mathbf{u}}{\partial t}$ , and  $\frac{\partial^2 \mathbf{u}}{\partial t^2}$  are obtained. To bound  $\frac{\partial u_i}{\partial x}$ , for all  $i$  and any  $(x, t)$ , consider an interval  $I = (a, a + \sqrt{\varepsilon_i})$  such that  $x \in I$ .

Then for some  $y \in I$  and  $t \in (0, T]$

$$\frac{\partial u_i}{\partial x}(y, t) = \frac{u_i(a + \sqrt{\varepsilon_i}, t) - u_i(a, t)}{\sqrt{\varepsilon_i}}$$

$$|\frac{\partial u_i}{\partial x}(y, t)| \leq C\varepsilon_i^{-\frac{1}{2}}\|\mathbf{u}\|. \quad (6)$$

Then for any  $x \in I$

$$\begin{aligned}\frac{\partial u_i}{\partial x}(x, t) &= \frac{\partial u_i}{\partial x}(y, t) + \int_y^x \frac{\partial^2 u_i(s, t)}{\partial x^2} ds \\ \frac{\partial u_i}{\partial x}(x, t) &= \frac{\partial u_i}{\partial x}(y, t) + \varepsilon_i^{-1} \int_y^x \left( \frac{\partial u_i(s, t)}{\partial t} - f_i(s, t) + \sum_{j=1}^n a_{ij}(s, t) u_j(s, t) \right) ds \\ \left| \frac{\partial u_i}{\partial x}(x, t) \right| &\leq \left| \frac{\partial u_i}{\partial x}(y, t) \right| + C \varepsilon_i^{-1} \int_y^x (||\mathbf{u}||_r + ||\mathbf{f}|| + \left\| \frac{\partial \mathbf{f}}{\partial t} \right\|) ds.\end{aligned}$$

Using (6) in the above equation

$$\left| \frac{\partial u_i}{\partial x}(x, t) \right| \leq C \varepsilon_i^{-\frac{1}{2}} (||\mathbf{u}||_r + ||\mathbf{f}|| + \left\| \frac{\partial \mathbf{f}}{\partial t} \right\|).$$

Rearranging the terms in (1), it is easy to get

$$\left| \frac{\partial^2 u_i}{\partial x^2} \right| \leq C \varepsilon_i^{-1} (||\mathbf{u}||_r + ||\mathbf{f}|| + \left\| \frac{\partial \mathbf{f}}{\partial t} \right\|).$$

Analogous steps are used to get the rest of the estimates. ■

The Shishkin decomposition of the exact solution  $\mathbf{u}$  of (1) is  $\mathbf{u} = \mathbf{v} + \mathbf{w}$  where the smooth component  $\mathbf{v}$  is the solution of

$$\mathbf{L}\mathbf{v} = \mathbf{f} \text{ in } \Omega, \mathbf{v} = \mathbf{u}_0 \text{ on } \Gamma \quad (7)$$

and the singular component  $\mathbf{w}$  is the solution of

$$\mathbf{L}\mathbf{w} = \mathbf{0} \text{ in } \Omega, \mathbf{w} = \mathbf{u} - \mathbf{v} \text{ on } \Gamma. \quad (8)$$

For convenience the left and right boundary layers of  $\mathbf{w}$  are separated using the further decomposition  $\mathbf{w} = \mathbf{w}^L + \mathbf{w}^R$  where  $\mathbf{L}\mathbf{w}^L = \mathbf{0}$  on  $\Omega$ ,  $\mathbf{w}^L = \mathbf{w}$  on  $\Gamma_L$ ,  $\mathbf{w}^L = \mathbf{0}$  on  $\Gamma_B \cup \Gamma_R$  and  $\mathbf{L}\mathbf{w}^R = \mathbf{0}$  on  $\Omega$ ,  $\mathbf{w}^R = \mathbf{w}$  on  $\Gamma_R$ ,  $\mathbf{w}^R = \mathbf{0}$  on  $\Gamma_L \cup \Gamma_B$ .

Bounds on the smooth component and its derivatives are contained in

**Lemma 4.** *Let  $A(x, t)$  satisfy (2) and (3). Then the smooth component  $\mathbf{v}$  and its derivatives satisfy, for all  $(x, t) \in \overline{\Omega}$  and each  $i = 1, \dots, n$ ,*

$$\begin{aligned}\left| \frac{\partial^l v_i}{\partial t^l}(x, t) \right| &\leq C \text{ for } l = 0, 1, 2 \\ \left| \frac{\partial^l v_i}{\partial x^l}(x, t) \right| &\leq C(1 + \varepsilon_i^{1-\frac{l}{2}}) \text{ for } l = 0, 1, 2, 3, 4 \\ \left| \frac{\partial^l v_i}{\partial x^{l-1} \partial t}(x, t) \right| &\leq C \text{ for } l = 2, 3.\end{aligned}$$

*Proof.* The bound on  $\mathbf{v}$  is an immediate consequence of the defining equations for  $\mathbf{v}$  and Lemma (2). Differentiating the equation (7) twice partially with respect to  $x$  and applying Lemma 2 for  $\frac{\partial^2 v_i}{\partial x^2}$ , we get

$$|\frac{\partial^2 v_i}{\partial x^2}| \leq C(1 + \|\frac{\partial \mathbf{v}}{\partial x}\|). \quad (9)$$

Let

$$\frac{\partial v_{i^*}}{\partial x}(x^*, t^*) = \|\frac{\partial \mathbf{v}}{\partial x}\| \quad \text{for some } i = i^*, x = x^*, t = t^*. \quad (10)$$

Using Taylor expansion, it follows that, for some  $y \in [0, 1 - x^*]$  and some  $\eta \in (x^*, x^* + y)$

$$v_{i^*}(x^* + y, t^*) = v_{i^*}(x^*, t^*) + y \frac{\partial v_{i^*}}{\partial x}(x^*, t^*) + \frac{y^2}{2} \frac{\partial^2 v_{i^*}}{\partial x^2}(\eta, t^*). \quad (11)$$

Rearranging (11) yields

$$\begin{aligned} \frac{\partial v_{i^*}}{\partial x}(x^*, t^*) &= \frac{v_{i^*}(x^* + y, t^*) - v_{i^*}(x^*, t^*)}{y} - \frac{y}{2} \frac{\partial^2 v_{i^*}}{\partial x^2}(\eta, t^*) \\ |\frac{\partial v_{i^*}}{\partial x}(x^*, t^*)| &\leq \frac{2}{y} \|\mathbf{v}\| + \frac{y}{2} \|\frac{\partial^2 \mathbf{v}}{\partial x^2}\|. \end{aligned} \quad (12)$$

Using (10) and (12) in (9),

$$|\frac{\partial^2 v_i}{\partial x^2}| \leq C(1 + \frac{2}{y} \|\mathbf{v}\| + \frac{y}{2} \|\frac{\partial^2 \mathbf{v}}{\partial x^2}\|).$$

This leads to

$$(1 - \frac{Cy}{2}) \|\frac{\partial^2 \mathbf{v}}{\partial x^2}\| \leq C(1 + \frac{2}{y} \|\mathbf{v}\|)$$

or

$$\|\frac{\partial^2 \mathbf{v}}{\partial x^2}\| \leq C. \quad (13)$$

Using (13) in (12) yields

$$\|\frac{\partial \mathbf{v}}{\partial x}\| \leq C.$$

Repeating the above steps with  $\frac{\partial v_i}{\partial t}$ , it is easy to get the required bounds on the mixed derivatives. The bounds on  $\frac{\partial^3 \mathbf{v}}{\partial x^3}$ ,  $\frac{\partial^4 \mathbf{v}}{\partial x^4}$  are derived by a similar argument.  $\blacksquare$

### 3 Improved estimates

The layer functions  $B_i^L$ ,  $B_i^R$ ,  $B_i$ ,  $i = 1, \dots, n$ , associated with the solution  $\mathbf{u}$ , are defined on  $[0, 1]$  by

$$B_i^L(x) = e^{-x\sqrt{\alpha/\varepsilon_i}}, \quad B_i^R(x) = B_i^L(1-x), \quad B_i(x) = B_i^L(x) + B_i^R(x).$$

The following elementary properties of these layer functions, for all  $1 \leq i < j \leq n$  and  $0 \leq x < y \leq 1$ , should be noted:

- (a)  $B_i^L(x) < B_j^L(x)$ ,  $B_i^L(x) > B_i^L(y)$ ,  $0 < B_i^L(x) \leq 1$ .
- (b)  $B_i^R(x) < B_j^R(x)$ ,  $B_i^R(x) < B_i^R(y)$ ,  $0 < B_i^R(x) \leq 1$ .
- (c)  $B_i(x)$  is monotone decreasing (increasing) for increasing  $x \in [0, \frac{1}{2}]$  ( $[\frac{1}{2}, 1]$ ).
- (d)  $B_i(x) \leq 2B_i^L(x)$  for  $x \in [0, \frac{1}{2}]$ ,  $B_i(x) \leq 2B_i^R(x)$  for  $x \in [\frac{1}{2}, 1]$ .

**Definition 1.** For  $B_i^L$ ,  $B_j^L$ , each  $i, j$ ,  $1 \leq i \neq j \leq n$  and each  $s, s > 0$ , the point  $x_{i,j}^{(s)}$  is defined by

$$\frac{B_i^L(x_{i,j}^{(s)})}{\varepsilon_i^s} = \frac{B_j^L(x_{i,j}^{(s)})}{\varepsilon_j^s}. \quad (14)$$

It is remarked that

$$\frac{B_i^R(1-x_{i,j}^{(s)})}{\varepsilon_i^s} = \frac{B_j^R(1-x_{i,j}^{(s)})}{\varepsilon_j^s}. \quad (15)$$

In the next lemma the existence and uniqueness of the points  $x_{i,j}^{(s)}$  are shown. Various properties are also established.

**Lemma 5.** For all  $i, j$ , such that  $1 \leq i < j \leq n$  and  $0 < s \leq 3/2$ , the points  $x_{i,j}$  exist, are uniquely defined and satisfy the following inequalities

$$\frac{B_i^L(x)}{\varepsilon_i^s} > \frac{B_j^L(x)}{\varepsilon_j^s}, \quad x \in [0, x_{i,j}^{(s)}), \quad \frac{B_i^L(x)}{\varepsilon_i^s} < \frac{B_j^L(x)}{\varepsilon_j^s}, \quad x \in (x_{i,j}^{(s)}, 1]. \quad (16)$$

Moreover

$$x_{i,j}^{(s)} < x_{i+1,j}^{(s)}, \quad \text{if } i+1 < j \quad \text{and} \quad x_{i,j}^{(s)} < x_{i,j+1}^{(s)}, \quad \text{if } i < j. \quad (17)$$

Also

$$x_{i,j}^{(s)} < 2s\sqrt{\frac{\varepsilon_j}{\alpha}} \quad \text{and} \quad x_{i,j}^{(s)} \in (0, \frac{1}{2}) \quad \text{if } i < j. \quad (18)$$

Analogous results hold for the  $B_i^R$ ,  $B_j^R$  and the points  $1 - x_{i,j}^{(s)}$ .

*Proof.* Existence, uniqueness and (16) follow from the observation that the ratio of the two sides of (14), namely

$$\frac{B_i^L(x)}{\varepsilon_i^s} \frac{\varepsilon_j^s}{B_j^L(x)} = \frac{\varepsilon_j^s}{\varepsilon_i^s} \exp(-\sqrt{\alpha}x(\frac{1}{\sqrt{\varepsilon_i}} - \frac{1}{\sqrt{\varepsilon_j}})),$$

is monotonically decreasing from the value  $\frac{\varepsilon_j^s}{\varepsilon_i^s} > 1$  as  $x$  increases from 0.

The point  $x_{i,j}^{(s)}$  is the unique point  $x$  at which this ratio has the value 1. Rearranging (14), and using the inequality  $\ln x < x - 1$  for all  $x > 1$ , gives

$$x_{i,j}^{(s)} = 2s \left[ \frac{\ln(\frac{1}{\sqrt{\varepsilon_i}}) - \ln(\frac{1}{\sqrt{\varepsilon_j}})}{\sqrt{\alpha}(\frac{1}{\sqrt{\varepsilon_i}} - \frac{1}{\sqrt{\varepsilon_j}})} \right] = \frac{2s \ln(\frac{\sqrt{\varepsilon_j}}{\sqrt{\varepsilon_i}})}{\sqrt{\alpha}(\frac{1}{\sqrt{\varepsilon_i}} - \frac{1}{\sqrt{\varepsilon_j}})} < 2s \sqrt{\frac{\varepsilon_j}{\alpha}}, \quad (19)$$

which is the first part of (19). The second part follows immediately from this and (4).

To prove (17), writing  $\sqrt{\varepsilon_k} = \exp(-p_k)$ , for some  $p_k > 0$  and all  $k$ , it follows that

$$x_{i,j}^{(s)} = \frac{2s(p_i - p_j)}{\sqrt{\alpha}(\exp p_i - \exp p_j)}.$$

The inequality  $x_{i,j}^{(s)} < x_{i+1,j}^{(s)}$  is equivalent to

$$\frac{p_i - p_j}{\exp p_i - \exp p_j} < \frac{p_{i+1} - p_j}{\exp p_{i+1} - \exp p_j},$$

which can be written in the form

$$(p_{i+1} - p_j) \exp(p_i - p_j) + (p_i - p_{i+1}) - (p_i - p_j) \exp(p_{i+1} - p_j) > 0.$$

With  $a = p_i - p_j$  and  $b = p_{i+1} - p_j$  it is not hard to see that  $a > b > 0$  and  $a - b = p_i - p_{i+1}$ . Moreover, the previous inequality is then equivalent to

$$\frac{\exp a - 1}{a} > \frac{\exp b - 1}{b},$$

which is true because  $a > b$  and proves the first part of (17). The second part is proved by a similar argument.

The analogous results for the  $B_i^R$ ,  $B_j^R$  and the points  $1 - x_{i,j}^{(s)}$  are proved by a similar argument. ■

In the following lemma sharper estimates of the smooth component are presented.

**Lemma 6.** *Let  $A(x, t)$  satisfy (2) and (3). Then the smooth component  $\mathbf{v}$  of the solution  $\mathbf{u}$  of (1) satisfies for all  $i = 1, \dots, n$  and all  $(x, t) \in \overline{\Omega}$*

$$|\frac{\partial^l v_i}{\partial x^l}(x, t)| \leq C \left( 1 + \sum_{q=i}^n \frac{B_q(x)}{\varepsilon_q^{\frac{l}{2}-1}} \right) \text{ for } l = 0, 1, 2, 3.$$

*Proof.* Define two barrier functions

$$\psi^\pm(x, t) = C[1 + B_n(x)]\mathbf{e} \pm \frac{\partial^l \mathbf{v}}{\partial x^l}(x, t), \quad l = 0, 1, 2 \quad \text{and} \quad (x, t) \in \overline{\Omega}.$$



We find that, for a proper choice of  $C$ ,

$$\begin{aligned}\psi_i^\pm(0, t) &= C \pm \frac{\partial^l u_{0,i}}{\partial x^l}(0, t) = C \geq 0 \\ \psi_i^\pm(1, t) &= C \pm \frac{\partial^l u_{0,i}}{\partial x^l}(1, t) = C \geq 0 \\ \psi_i^\pm(x, 0) &= C[1 + B_n(x)] \pm \frac{\partial^l \phi_{B,i}(x)}{\partial x^l} = C[1 + B_n(x)] \pm C \geq 0\end{aligned}$$

as  $\phi_b(x) \in C^{(2)}(\Gamma_b)$  and  $(\mathbf{L}\psi^\pm)_i(x, t) \geq 0$ .

Using Lemma 1, we conclude that

$$|\frac{\partial^l v_i}{\partial x^l}(x, t)| \leq C[1 + B_n(x)] \text{ for } l = 0, 1, 2. \quad (20)$$

Consider the equation

$$(\mathbf{L}(\frac{\partial^2 \mathbf{v}}{\partial x^2}))_i = \frac{\partial^2 f_i}{\partial x^2} - 2 \frac{\partial \sum_{j=1}^n a_{ij}}{\partial x} \frac{\partial v_j}{\partial x} - \frac{\partial^2 \sum_{j=1}^n a_{ij}}{\partial x^2} v_j \quad (21)$$

with

$$\frac{\partial^2 v_i}{\partial x^2}(0, t) = 0, \frac{\partial^2 v_i}{\partial x^2}(1, t) = 0, \frac{\partial^2 v_i}{\partial x^2}(x, 0) = \frac{\partial^2 \phi_{b,i}(x)}{\partial x^2}. \quad (22)$$

For convenience, let  $\mathbf{p}$  denote  $\frac{\partial^2 \mathbf{v}}{\partial x^2}$ . Then

$$\mathbf{Lp} = \mathbf{g} \text{ with } \mathbf{p}(0, t) = \mathbf{0}, \mathbf{p}(1, t) = \mathbf{0}, \mathbf{p}(x, 0) = \mathbf{s} \quad (23)$$

where

$$g_i = \frac{\partial^2 f_i}{\partial x^2} - 2 \frac{\partial \sum_{j=1}^n a_{ij}}{\partial x} \frac{\partial v_j}{\partial x} - \sum_{j=1}^n \frac{\partial^2 a_{ij}}{\partial x^2} v_j \text{ and } s_i = \frac{\partial^2 \phi_{b,i}}{\partial x^2}(x).$$

Let  $\mathbf{q}$  and  $\mathbf{r}$  be the smooth and singular components of  $\mathbf{p}$  given by

$$\mathbf{Lq} = \mathbf{g} \text{ with } \mathbf{q}(0, t) = \mathbf{p}_0(0, t), \mathbf{q}(1, t) = \mathbf{p}_0(1, t), \mathbf{q}(x, 0) = \mathbf{p}(x, 0) \quad (24)$$

where  $\mathbf{p}_0$  is the solution of the reduced problem

$$\frac{\partial \mathbf{p}_0}{\partial t} + A\mathbf{p}_0 = \mathbf{g} \text{ with } \mathbf{p}_0(x, 0) = \mathbf{p}(x, 0) = \mathbf{s}.$$

Now,

$$\mathbf{Lr} = \mathbf{0}, \text{ with } \mathbf{r}(0, t) = -\mathbf{q}(0, t), \mathbf{r}(1, t) = -\mathbf{q}(1, t), \mathbf{r}(x, 0) = \mathbf{0}. \quad (25)$$

Using Lemma 4 and Lemma 7, we have for  $i = 1, \dots, n$  and  $(x, t) \in \overline{\Omega}$

$$|\frac{\partial q_i}{\partial x}(x, t)| \leq C$$

and

$$|\frac{\partial r_i}{\partial x}(x, t)| \leq C[\frac{B_i(x)}{\sqrt{\varepsilon_i}} + \dots + \frac{B_n(x)}{\sqrt{\varepsilon_n}}].$$

Hence, for  $(x, t) \in \overline{\mathcal{O}}$  and  $i = 1, \dots, n$ ,

$$|\frac{\partial^3 v_i}{\partial x^3}| = |\frac{\partial p_i}{\partial x}| \leq C[1 + \frac{B_i(x)}{\sqrt{\varepsilon_i}} + \dots + \frac{B_n(x)}{\sqrt{\varepsilon_n}}]. \quad (26)$$

From (20) and (26), we find that for  $l = 0, 1, 2, 3$  and  $(x, t) \in \overline{\mathcal{O}}$

$$|\frac{\partial^l v_i}{\partial x^l}| \leq C[1 + \varepsilon_i^{1-\frac{l}{2}} B_i(x) + \dots + \varepsilon_i^{1-\frac{l}{2}} B_n(x)]. \quad \blacksquare$$

**Remark :** It is interesting to note that the above estimate reduces to the estimate of the smooth component of the solution of the scalar problem given in [1] when  $n = 1$ .

Bounds on the singular components  $\mathbf{w}^L$ ,  $\mathbf{w}^R$  of  $\mathbf{u}$  and their derivatives are contained in

**Lemma 7.** *Let  $A(x, t)$  satisfy (2) and (3). Then there exists a constant  $C$ , such that, for each  $(x, t) \in \overline{\mathcal{O}}$  and  $i = 1, \dots, n$ ,*

$$|\frac{\partial^l w_i^L}{\partial t^l}(x, t)| \leq C B_n^L(x), \text{ for } l = 0, 1, 2.$$

$$|\frac{\partial^l w_i^L}{\partial x^l}(x, t)| \leq C \sum_{q=i}^n \frac{B_q^L(x)}{\varepsilon_q^{\frac{l}{2}}}, \text{ for } l = 1, 2.$$

$$|\frac{\partial^3 w_i^L}{\partial x^3}(x, t)| \leq C \sum_{q=1}^n \frac{B_q^L(x)}{\varepsilon_q^{\frac{3}{2}}}.$$

$$|\frac{\partial^4 w_i^L}{\partial x^4}(x, t)| \leq C \frac{1}{\varepsilon_i} \sum_{q=1}^n \frac{B_q^L(x)}{\varepsilon_q}.$$

Analogous results hold for  $w_i^R$  and its derivatives.

*Proof.* To obtain the bound of  $\mathbf{w}^L$ , define the functions  $\psi_i^\pm(x, t) = Ce^{\alpha t} B_n^L(x) \pm w_i^L(x, t)$ , for each  $i = 1, \dots, n$ . It is clear that  $\psi_i^\pm(0, t)$ ,  $\psi_i^\pm(x, 0)$ ,  $\psi_i^\pm(1, t)$  and  $(\mathbf{L}\psi^\pm)_i(x, t)$  are non-negative. By Lemma 1,  $\psi_i^\pm(x, t) \geq 0$ . It follows that  $|w_i^L| \leq Ce^{\alpha t} B_n^L(x)$

or

$$|w_i^L| \leq C B_n^L(x). \quad (27)$$

To obtain the bound for  $\frac{\partial w_i^L}{\partial t}$ , define the two functions  $\theta_i^\pm(x, t) = CB_n^L(x) \pm \frac{\partial w_i^L}{\partial t}(x, t)$  for each  $i = 1, \dots, n$ . Differentiating the homogeneous equation satisfied by  $w_i^L$ , partially with respect to  $t$ , and rearranging yields

$$\frac{\partial^2 w_i^L}{\partial t^2} - \varepsilon_i \frac{\partial^3 w_i^L}{\partial x^2 \partial t} + \sum_{j=1}^n a_{ij} \frac{\partial w_j^L}{\partial t} = \frac{-\partial \sum_{j=1}^n a_{ij}}{\partial t} w_j^L,$$

and we get

$$\begin{aligned} |L \frac{\partial w_i^L}{\partial t}| &\leq CB_n^L(x) \\ |\frac{\partial w_i^L}{\partial t}(0, t)| &\leq |\frac{\partial u_i}{\partial t}(0, t)| + |\frac{\partial v_i}{\partial t}(0, t)| = |\frac{\partial \phi_{L,i}(t)}{\partial t}| \leq C \\ |\frac{\partial w_i^L}{\partial t}(1, t)| &\leq |\frac{\partial u_i}{\partial t}(1, t)| + |\frac{\partial v_i}{\partial t}(1, t)| = |\frac{\partial \phi_{R,i}(t)}{\partial t}| \leq C \\ |\frac{\partial w_i^L}{\partial t}(x, 0)| &\leq |\frac{\partial \phi_{B,i}(x)}{\partial t}| = 0. \end{aligned}$$

By Lemma 2, it follows that

$$|\frac{\partial w_i^L}{\partial t}| \leq CB_n^L(x). \quad (28)$$

Now the bound for  $\frac{\partial^2 w_i^L}{\partial x \partial t}$  is obtained by using Lemma (3) and Lemma (4)

$$\begin{aligned} |\frac{\partial^2 w_i^L}{\partial x \partial t}| &\leq |\frac{\partial^2 u_i}{\partial x \partial t}| + |\frac{\partial^2 v_i}{\partial x \partial t}| \\ |\frac{\partial^2 w_i^L}{\partial x \partial t}| &\leq C\varepsilon_i^{-\frac{1}{2}}(\|\mathbf{u}\|_r + \|\mathbf{f}\| + \|\frac{\partial \mathbf{f}}{\partial t}\| + \|\frac{\partial^2 \mathbf{f}}{\partial t^2}\|). \end{aligned}$$

Similarly,

$$|\frac{\partial^3 w_i^L}{\partial x^2 \partial t}| \leq C\varepsilon_i^{-1}(\|\mathbf{u}\|_r + \|\mathbf{f}\| + \|\frac{\partial \mathbf{f}}{\partial t}\| + \|\frac{\partial^2 \mathbf{f}}{\partial t^2}\|).$$

The bounds on  $\frac{\partial^l w_i^L}{\partial x^l}$ ,  $l = 1, 2, 3, 4$  and  $i = 1, \dots, n$  are derived by the method of induction on  $n$ . It is assumed that the bounds  $\frac{\partial w_i^L}{\partial x}$ ,  $\frac{\partial^2 w_i^L}{\partial x^2}$ ,  $\frac{\partial^3 w_i^L}{\partial x^3}$  and  $\frac{\partial^4 w_i^L}{\partial x^4}$  hold for all systems up to  $n - 1$ . Define  $\tilde{\mathbf{w}}^L = (w_1^L, \dots, w_{n-1}^L)$ , then  $\tilde{\mathbf{w}}^L$  satisfies the system

$$\frac{\partial \tilde{\mathbf{w}}^L}{\partial t} - \tilde{E} \frac{\partial^2 \tilde{\mathbf{w}}^L}{\partial x^2} + \tilde{A} \tilde{\mathbf{w}}^L = \mathbf{g},$$

with

$$\tilde{\mathbf{w}}^L(0, t) = \tilde{\mathbf{u}}(0, t) - \tilde{\mathbf{u}}_0(0, t), \tilde{\mathbf{w}}^L(1, t) = \tilde{\mathbf{0}},$$

$$\tilde{\mathbf{w}}^L(x, 0) = \tilde{\mathbf{u}}(x, 0) - \tilde{\mathbf{u}}_0(x, 0) = \tilde{\phi}_B(x) - \tilde{\phi}_B(x) = \tilde{\mathbf{0}}.$$

Here,  $\tilde{E}$  and  $\tilde{A}$  are the matrices obtained by deleting the last row and column from  $E, A$  respectively, the components of  $\mathbf{g}$  are  $g_i = -a_{i,n}w_n^L$  for  $1 \leq i \leq n-1$  and  $\tilde{\mathbf{u}}_0$  is the solution of the reduced problem. Now decompose  $\tilde{\mathbf{w}}^L$  into smooth and singular components to get  $\tilde{\mathbf{w}}^L = \mathbf{q} + \mathbf{r}$ ,  $\frac{\partial \tilde{\mathbf{w}}^L}{\partial x} = \frac{\partial \mathbf{q}}{\partial x} + \frac{\partial \mathbf{r}}{\partial x}$ . By induction, the bounds on the derivatives of  $\tilde{\mathbf{w}}^L$  hold. That is for  $i = 1, \dots, n-1$

$$\left. \begin{aligned} \left| \frac{\partial w_i^L}{\partial x} \right| &\leq C \sum_{q=i}^{n-1} \varepsilon_q^{-\frac{1}{2}} B_q^L(x) \\ \left| \frac{\partial^2 w_i^L}{\partial x^2} \right| &\leq C \sum_{q=i}^{n-1} \varepsilon_q^{-1} B_q^L(x) \\ \left| \frac{\partial^3 w_i^L}{\partial x^3} \right| &\leq C \sum_{q=1}^{n-1} \varepsilon_q^{-\frac{3}{2}} B_q^L(x) \\ \left| \varepsilon_i \frac{\partial^4 w_i^L}{\partial x^4} \right| &\leq C \sum_{q=1}^{n-1} \varepsilon_q^{-1} B_q^L(x) \end{aligned} \right\} \quad (29)$$

Rearranging the  $n^{th}$  equation of the system satisfied by  $w_n^L$  yields

$$\varepsilon_n \frac{\partial^2 w_n^L}{\partial x^2} = \frac{\partial w_n^L}{\partial t} + \sum_{j=1}^n a_{nj} w_j^L.$$

Using (27) and (28) gives

$$\left| \frac{\partial^2 w_n^L}{\partial x^2} \right| \leq C \varepsilon_n^{-1} B_n^L(x). \quad (30)$$

Applying the mean value theorem to  $w_n^L$  at some  $y$ ,  $a < y < a + \sqrt{\varepsilon_n}$

$$\frac{\partial w_n^L}{\partial x}(y, t) = \frac{w_n^L(a + \sqrt{\varepsilon_n}, t) - w_n^L(a, t)}{\sqrt{\varepsilon_n}}$$

Using (27) gives

$$\left| \frac{\partial w_n^L}{\partial x}(y, t) \right| \leq \frac{C}{\sqrt{\varepsilon_n}} (B_n^L(a + \sqrt{\varepsilon_n}) + B_n^L(a)).$$

So

$$\left| \frac{\partial w_n^L}{\partial x}(y, t) \right| \leq \frac{C}{\sqrt{\varepsilon_n}} B_n^L(a). \quad (31)$$

Again

$$\frac{\partial w_n^L}{\partial x}(x, t) = \frac{\partial w_n^L}{\partial x}(y, t) + (y - x) \frac{\partial^2 w_n^L}{\partial x^2}(\eta, t), \quad y < \eta < x. \quad (32)$$

Using (30) and (31) in (32) yields

$$\begin{aligned}
\left| \frac{\partial w_n^L}{\partial x}(x, t) \right| &\leq C[\varepsilon_n^{\frac{-1}{2}} B_n^L(a) + \varepsilon_n^{\frac{-1}{2}} B_n^L(\eta)] \\
&\leq C\varepsilon_n^{\frac{-1}{2}} B_n^L(a) \\
&= C\varepsilon_n^{\frac{-1}{2}} B_n^L(x) \frac{B_n^L(a)}{B_n^L(x)} \\
&= C\varepsilon_n^{\frac{-1}{2}} B_n^L(x) e^{(x-a)\sqrt{\alpha}/\sqrt{\varepsilon_n}} \\
&= C\varepsilon_n^{\frac{-1}{2}} B_n^L(x) e^{\sqrt{\varepsilon_n}\sqrt{\alpha}/\sqrt{\varepsilon_n}}.
\end{aligned}$$

Therefore

$$\left| \frac{\partial w_n^L}{\partial x}(x, t) \right| \leq C\varepsilon_n^{\frac{-1}{2}} B_n^L(x). \quad (33)$$

Now, differentiating the equation satisfied by  $w_n^L$  partially with respect to  $x$ , and rearranging, gives

$$\varepsilon_n \frac{\partial^3 w_n^L}{\partial x^3} = \frac{\partial^2 w_n^L}{\partial x \partial t} + \sum_{q=1}^{n-1} a_{nq} \frac{\partial w_q^L}{\partial x} + a_{nn} \frac{\partial w_n^L}{\partial x} + \sum_{q=1}^n \frac{\partial a_{nq}}{\partial x} w_q^L.$$

The bounds on  $w_n^L$  and (29) then give

$$\left| \frac{\partial^3 w_n^L}{\partial x^3} \right| \leq C \sum_{q=1}^n \varepsilon_q^{\frac{-3}{2}} B_q^L(x).$$

Similarly

$$\left| \varepsilon_n \frac{\partial^4 w_n^L}{\partial x^4} \right| \leq C \sum_{q=1}^n \varepsilon_q^{-1} B_q^L(x).$$

Using the bounds on  $w_n^L$ ,  $\frac{\partial w_n^L}{\partial x}$ ,  $\frac{\partial^2 w_n^L}{\partial x^2}$ ,  $\frac{\partial^3 w_n^L}{\partial x^3}$  and  $\frac{\partial^4 w_n^L}{\partial x^4}$ , it is seen that  $\mathbf{g}$ ,  $\frac{\partial \mathbf{g}}{\partial x}$ ,  $\frac{\partial^2 \mathbf{g}}{\partial x^2}$ ,  $\frac{\partial^3 \mathbf{g}}{\partial x^3}$ ,  $\frac{\partial^4 \mathbf{g}}{\partial x^4}$  are bounded by  $CB_n^L(x)$ ,  $C \frac{B_n^L(x)}{\sqrt{\varepsilon_n}}$ ,  $C \frac{B_n^L(x)}{\varepsilon_n}$ ,  $\sum_{q=1}^n \frac{B_q^L(x)}{\varepsilon_q^{\frac{3}{2}}}$ ,  $C\varepsilon_n^{-1} \sum_{q=1}^n \frac{B_q^L(x)}{\varepsilon_q}$  respectively. Introducing the functions  $\psi^\pm(x, t) = CB_n^L(x)\mathbf{e} \pm \mathbf{q}(x, t)$ , it is easy to see that  $\psi^\pm(0, t) = C\mathbf{e} \pm \mathbf{q}(0, t) \geq \mathbf{0}$ ,  $\psi^\pm(1, t) = CB_n^L(1)\mathbf{e} \pm \mathbf{0} \geq \mathbf{0}$ ,  $\psi^\pm(x, 0) = CB_n^L(x)\mathbf{e} \pm \mathbf{0} \geq \mathbf{0}$  and

$$(\mathbf{L}\psi^\pm)_i(x, t) = C(-\varepsilon_i \frac{\alpha}{\varepsilon_n} + \sum_{j=1}^n a_{ij})B_n^L(x) \pm CB_n^L(x) \geq 0.$$

Applying Lemma 1, it follows that  $\|\mathbf{q}(x, t)\| \leq CB_n^L(x)$ . Defining barrier functions  $\theta^\pm(x, t) = C\varepsilon_n^{\frac{-1}{2}} B_n^L(x)\mathbf{e} \pm \frac{\partial \mathbf{q}}{\partial x}$  and using Lemma 3 for the problem satisfied by  $\mathbf{q}$ , the bound required for  $\frac{\partial \mathbf{q}}{\partial x}$  and  $\frac{\partial^2 \mathbf{q}}{\partial x^2}$  is obtained. By induction, the following bounds for  $\mathbf{r}$  are obtained for  $i = 1, \dots, n-1$ ,

$$\begin{aligned}
\left| \frac{\partial r_i}{\partial x} \right| &\leq \left[ \frac{B_i^L(x)}{\sqrt{\varepsilon_i}} + \cdots + \frac{B_{n-1}^L(x)}{\sqrt{\varepsilon_{n-1}}} \right], \\
\left| \frac{\partial^2 r_i}{\partial x^2} \right| &\leq C \left[ \frac{B_i^L(x)}{\varepsilon_i} + \cdots + \frac{B_{n-1}^L(x)}{\varepsilon_{n-1}} \right], \\
\left| \frac{\partial^3 r_i}{\partial x^3} \right| &\leq C \left[ \frac{B_1^L(x)}{\varepsilon_1^{\frac{3}{2}}} + \cdots + \frac{B_{n-1}^L(x)}{\varepsilon_{n-1}^{\frac{3}{2}}} \right], \\
|\varepsilon_i \frac{\partial^4 r_i}{\partial x^4}| &\leq C \left[ \frac{B_1^L(x)}{\varepsilon_1} + \cdots + \frac{B_{n-1}^L(x)}{\varepsilon_{n-1}} \right].
\end{aligned}$$

Combining the bounds for the derivatives of  $q_i$  and  $r_i$  it follows that, for  $i = 1, 2, \dots, n$

$$\begin{aligned}
\left| \frac{\partial^l w_i^L}{\partial x^l} \right| &\leq \left| \frac{\partial^l q_i}{\partial x^l} \right| + \left| \frac{\partial^l r_i}{\partial x^l} \right| \\
\left| \frac{\partial^l w_i^L}{\partial x^l} \right| &\leq C \sum_{q=i}^n \frac{B_q^L(x)}{\varepsilon_q^{\frac{l}{2}}} \quad \text{for } l = 1, 2 \\
\left| \frac{\partial^3 w_i^L}{\partial x^3} \right| &\leq C \sum_{q=1}^n \frac{B_q^L(x)}{\varepsilon_q^{\frac{3}{2}}} \\
\text{and } |\varepsilon_i \frac{\partial^4 w_i^L}{\partial x^4}| &\leq C \sum_{q=1}^n \frac{B_q^L(x)}{\varepsilon_q}.
\end{aligned}$$

Recalling the bounds on the derivatives of  $w_n^L$  completes the proof of the lemma for the system of  $n$  equations.

A similar proof of the analogous results for the right boundary layer functions holds. ■

## 4 The Shishkin mesh

A piecewise uniform Shishkin mesh with  $M \times N$  mesh-intervals is now constructed. Let  $\Omega_t^M = \{t_k\}_{k=1}^M$ ,  $\Omega_x^N = \{x_j\}_{j=1}^{N-1}$ ,  $\overline{\Omega}_t^M = \{t_k\}_{k=0}^M$ ,  $\overline{\Omega}_x^N = \{x_j\}_{j=0}^N$ ,  $\Omega^{M,N} = \Omega_t^M \times \Omega_x^N$ ,  $\overline{\Omega}^{M,N} = \overline{\Omega}_t^M \times \overline{\Omega}_x^N$  and  $\Gamma^{M,N} = \Gamma \cap \overline{\Omega}^{M,N}$ . The mesh  $\overline{\Omega}_t^M$  is chosen to be a uniform mesh with  $M$  mesh-intervals on  $[0, T]$ . The mesh  $\overline{\Omega}_x^N$  is a piecewise-uniform mesh on  $[0, 1]$  obtained by dividing  $[0, 1]$  into  $2n + 1$  mesh-intervals as follows

$$[0, \sigma_1] \cup \cdots \cup (\sigma_{n-1}, \sigma_n] \cup (\sigma_n, 1 - \sigma_n] \cup (1 - \sigma_n, 1 - \sigma_{n-1}] \cup \cdots \cup (1 - \sigma_1, 1].$$

The  $n$  parameters  $\sigma_r$ , which determine the points separating the uniform meshes, are defined by

$$\sigma_n = \min \left\{ \frac{1}{4}, 2\sqrt{\frac{\varepsilon_n}{\alpha}} \ln N \right\} \quad (34)$$

and for  $r = 1, \dots, n-1$

$$\sigma_r = \min \left\{ \frac{\sigma_{r+1}}{2}, 2\sqrt{\frac{\varepsilon_r}{\alpha}} \ln N \right\}. \quad (35)$$

Clearly

$$0 < \sigma_1 < \dots < \sigma_n \leq \frac{1}{4}, \quad \frac{3}{4} \leq 1 - \sigma_n < \dots < 1 - \sigma_1 < 1.$$

Then, on the sub-interval  $(\sigma_n, 1 - \sigma_n]$  a uniform mesh with  $\frac{N}{2}$  mesh-intervals is placed, on each of the sub-intervals  $(\sigma_r, \sigma_{r+1}]$  and  $(1 - \sigma_{r+1}, 1 - \sigma_r]$ ,  $r = 1, \dots, n-1$ , a uniform mesh of  $\frac{N}{2^{n-r+2}}$  mesh-intervals is placed and on both of the sub-intervals  $[0, \sigma_1]$  and  $(1 - \sigma_1, 1]$  a uniform mesh of  $\frac{N}{2^{n+1}}$  mesh-intervals is placed. In practice it is convenient to take

$$N = 2^{n+p+1} \quad (36)$$

for some natural number  $p$ . It follows that in the sub-interval  $[\sigma_{r-1}, \sigma_r]$  there are  $N/2^{n-r+3} = 2^{r+p-2}$  mesh-intervals. This construction leads to a class of  $2^n$  piecewise uniform Shishkin meshes  $\Omega^{M,N}$ . Note that these meshes are not the same as those constructed in [5]

The following notation is introduced:  $h_j = x_j - x_{j-1}$ ,  $J = \{x_j : D^+h_j = h_{j+1} - h_j \neq 0\}$ . Clearly,  $J$  is the set of points at which the mesh-size changes. Let  $R = \{r : \sigma_r \in J\}$ . From the above construction it follows that  $J$  is a subset of the set of transition points  $\{\sigma_r\}_{r=1}^n \cup \{1 - \sigma_r\}_{r=1}^n$ . It is not hard to see that for each point  $x_j$  in the mesh-interval  $(\sigma_{r-1}, \sigma_r]$ ,

$$h_j = 2^{n-r+3} N^{-1} (\sigma_r - \sigma_{r-1}) \quad (37)$$

and so the change in the mesh-size at the point  $\sigma_r$  is

$$D^+h_r = 2^{n-r+3} (d_r - d_{r-1}), \quad (38)$$

where  $d_r = \frac{\sigma_{r+1}}{2} - \sigma_r$  for  $1 \leq r \leq n$ , with the conventions  $d_0 = 0$ ,  $\sigma_{n+1} = 1/2$ . Notice that  $d_r \geq 0$ , that  $\Omega^{M,N}$  is a classical uniform mesh when  $d_r = 0$  for all  $r = 1 \dots n$  and, from (38), that

$$D^+h_r < 0 \text{ if } d_r = 0. \quad (39)$$

Furthermore

$$\sigma_r \leq C\sqrt{\varepsilon_r} \ln N, \quad 1 \leq r \leq n, \quad (40)$$

and, using (37), (40),

$$h_r + h_{r+1} \leq CN^{-1} \ln N \begin{cases} \sqrt{\varepsilon_{r+1}}, & \text{if } D^+ h_r > 0, \\ \sqrt{\varepsilon_r}, & \text{if } D^+ h_r < 0. \end{cases} \quad (41)$$

Also

$$\sigma_r = 2^{-(s-r+1)} \sigma_{s+1} \text{ when } d_r = \dots = d_s = 0, \ 1 \leq r \leq s \leq n. \quad (42)$$

The geometrical results in the following lemma are used later.

**Lemma 8.** *Assume that  $d_r > 0$  and let  $0 < s \leq 2$ . Then the following inequalities hold*

$$B_r^L(\sigma_r) = N^{-2}. \quad (43)$$

$$x_{r-1,r}^{(s)} \leq \sigma_r - h_r \text{ for } 1 < r \leq n. \quad (44)$$

$$\frac{B_q^L(\sigma_r)}{\varepsilon_q^s} \leq \frac{1}{\varepsilon_r^s} \text{ for } 1 \leq q \leq n, \ 1 \leq r \leq n. \quad (45)$$

$$B_q^L(\sigma_r - h_r) \leq CB_q^L(\sigma_r) \text{ for } 1 \leq r \leq q \leq n. \quad (46)$$

*Proof.* The proof of (43) follows immediately from the definition of  $\sigma_r$  and the assumption that  $d_r > 0$ .

To verify (44) note that, by Lemma 5 and (36),

$$x_{r-1,r}^{(s)} < 2s \sqrt{\frac{\varepsilon_r}{\alpha}} = \frac{s\sigma_r}{\ln N} = \frac{s\sigma_r}{(n+p+1) \ln 2} \leq \frac{\sigma_r}{2}.$$

Also, by (36) and (37),

$$h_r = 2^{n-r+3} N^{-1} (\sigma_r - \sigma_{r-1}) = 2^{2-r-p} (\sigma_r - \sigma_{r-1}) \leq \frac{\sigma_r - \sigma_{r-1}}{2} < \frac{\sigma_r}{2}.$$

It follows that  $x_{r-1,r}^{(s)} + h_r \leq \sigma_r$  as required.

To verify (45) note that if  $q \geq r$  the result is trivial. On the other hand, if  $q < r$ , by (44) and Lemma 5,

$$\frac{B_q^L(\sigma_r)}{\varepsilon_q^s} \leq \frac{B_q^L(x_{q,r}^{(s)})}{\varepsilon_q^s} = \frac{B_r^L(x_{q,r}^{(s)})}{\varepsilon_r^s} \leq \frac{1}{\varepsilon_r^s}.$$

Finally, to verify (46) note, from (37), that

$$h_r = 2^{n-r+3} N^{-1} (\sigma_r - \sigma_{r-1}) \leq 2^{n-r+3} N^{-1} \sigma_r = 2^{n-r+4} \sqrt{\frac{\varepsilon_r}{\alpha}} N^{-1} \ln N.$$

But

$$e^{2^{n-r+4} N^{-1} \ln N} = (N^{\frac{1}{N}})^{2^{n-r+4}} \leq C,$$

so



$$\sqrt{\frac{\alpha}{\varepsilon_q}} h_r \leq \sqrt{\frac{\varepsilon_r}{\varepsilon_q}} 2^{n-r+4} N^{-1} \ln N \leq 2^{n-r+4} N^{-1} \ln N \leq C,$$

since  $r \leq q$ . It follows that

$$B_q^L(\sigma_r - h_r) = B_q^L(\sigma_r) e^{\sqrt{\frac{\alpha}{\varepsilon_q}} h_r} \leq C B_q^L(\sigma_r)$$

as required. ■

## 5 The discrete problem

In this section a classical finite difference operator with an appropriate Shishkin mesh is used to construct a numerical method for (1), which is shown later to be essentially second order parameter-uniform. It is assumed henceforth that the problem data satisfy whatever smoothness conditions are required.

The discrete initial-boundary value problem is now defined on any mesh by the finite difference method

$$D_t^- \mathbf{U} - E \delta_x^2 \mathbf{U} + A \mathbf{U} = \mathbf{f} \text{ on } \Omega^{M,N}, \quad \mathbf{U} = \mathbf{u} \text{ on } \Gamma^{M,N}. \quad (47)$$

This is used to compute numerical approximations to the exact solution of (1). Note that (47), can also be written in the operator form

$$\mathbf{L}^{M,N} \mathbf{U} = \mathbf{f} \text{ on } \Omega^{M,N}, \quad \mathbf{U} = \mathbf{u} \text{ on } \Gamma^{M,N},$$

where

$$\mathbf{L}^{M,N} = D_t^- - E \delta_x^2 + A$$

and  $D_t^-$ ,  $\delta_x^2$ ,  $D_x^+$  and  $D_x^-$  are the difference operators

$$\begin{aligned} D_t^- \mathbf{U}(x_j, t_k) &= \frac{\mathbf{U}(x_j, t_k) - \mathbf{U}(x_j, t_{k-1})}{t_k - t_{k-1}}, \\ \delta_x^2 \mathbf{U}(x_j, t_k) &= \frac{D_x^+ \mathbf{U}(x_j, t_k) - D_x^- \mathbf{U}(x_j, t_k)}{(x_{j+1} - x_{j-1})/2}, \\ D_x^+ \mathbf{U}(x_j, t_k) &= \frac{\mathbf{U}(x_{j+1}, t_k) - \mathbf{U}(x_j, t_k)}{x_{j+1} - x_j}, \\ D_x^- \mathbf{U}(x_j, t_k) &= \frac{\mathbf{U}(x_j, t_k) - \mathbf{U}(x_{j-1}, t_k)}{x_j - x_{j-1}}. \end{aligned}$$

The following discrete results are analogous to those for the continuous case.

**Lemma 9.** *Let  $A(x, t)$  satisfy (2) and (3). Then, for any mesh function  $\Psi$ , the inequalities  $\Psi \geq \mathbf{0}$  on  $\Gamma^{M,N}$  and  $\mathbf{L}^{M,N}\Psi \geq \mathbf{0}$  on  $\Omega^{M,N}$  imply that  $\Psi \geq \mathbf{0}$  on  $\overline{\Omega}^{M,N}$ .*

*Proof.* Let  $i^*, j^*, k^*$  be such that  $\Psi_{i^*}(x_{j^*}, t_{k^*}) = \min_i \min_{j,k} \Psi_i(x_j, t_k)$  and assume that the lemma is false. Then  $\Psi_{i^*}(x_{j^*}, t_{k^*}) < 0$ . From the hypotheses we have  $j^* \neq 0$ ,  $N$  and  $\Psi_{i^*}(x_{j^*}, t_{k^*}) - \Psi_{i^*}(x_{j^*-1}, t_{k^*}) \leq 0$ ,  $\Psi_{i^*}(x_{j^*+1}, t_{k^*}) - \Psi_{i^*}(x_{j^*}, t_{k^*}) \geq 0$ , so  $\delta_x^2 \Psi_{i^*}(x_{j^*}, t_{k^*}) > 0$ . It follows that

$$(\mathbf{L}^N \Psi(x_{j^*}, t_{k^*}))_{i^*} = -\varepsilon_{i^*} \delta_x^2 \Psi_{i^*}(x_{j^*}, t_{k^*}) + \sum_{q=1}^n a_{i^*, q}(x_{j^*}, t_{k^*}) \Psi_q(x_{j^*}, t_{k^*}) < 0,$$

which is a contradiction, as required. ■

An immediate consequence of this is the following discrete stability result.

**Lemma 10.** *Let  $A(x, t)$  satisfy (2) and (3). Then, for any mesh function  $\Psi$  on  $\Omega$ ,*

$$\|\Psi(x_j, t_k)\| \leq \max \left\{ \|\Psi\|_{\Gamma^{M,N}}, \frac{1}{\alpha} \|\mathbf{L}^{M,N}\Psi\| \right\}.$$

*Proof.* Define the two functions

$$\Theta^\pm(x_j, t_k) = \max \left\{ \|\Psi\|_{\Gamma^{M,N}}, \frac{1}{\alpha} \|\mathbf{L}^{M,N}\Psi\| \right\} \mathbf{e} \pm \Psi(x_j, t_k)$$

where  $\mathbf{e} = (1, \dots, 1)$  is the unit vector. Using the properties of  $A$  it is not hard to verify that  $\Theta^\pm \geq \mathbf{0}$  on  $\Gamma^{M,N}$  and  $\mathbf{L}^{M,N}\Theta^\pm \geq \mathbf{0}$  on  $\Omega^{M,N}$ . It follows from Lemma 9 that  $\Theta^\pm \geq \mathbf{0}$  on  $\overline{\Omega}^{M,N}$ . ■

The following comparison result will be used in the proof of the error estimate.

**Lemma 11.** *(Comparison Principle) Assume that, for each  $i = 1, \dots, n$ , the mesh functions  $\Phi$  and  $\mathbf{Z}$  satisfy*

$$|Z_i| \leq \Phi_i, \text{ on } \Gamma^{M,N} \text{ and } |(\mathbf{L}^{M,N}\mathbf{Z})_i| \leq (\mathbf{L}^{M,N}\Phi)_i \text{ on } \Omega^{M,N}.$$

*Then, for each  $i = 1, \dots, n$ ,*

$$|Z_i| \leq \Phi_i.$$

*Proof.* Define the two mesh functions  $\Psi^\pm$  by

$$\Psi^\pm = \Phi \pm \mathbf{Z}.$$

Then, for each  $i = 1, \dots, n$ , satisfies

$$(\psi^\pm)_i \geq 0, \text{ on } \Gamma^{M,N} \text{ and } |(\mathbf{L}^{M,N}\mathbf{Z})_i| \leq (\mathbf{L}^{M,N}\Phi)_i \text{ on } \Omega^{M,N}.$$

The result follows from an application of Lemma 9. ■

## 6 The local truncation error

From Lemma 10, it is seen that in order to bound the error  $\|\mathbf{U} - \mathbf{u}\|$  it suffices to bound  $\mathbf{L}^{M,N}(\mathbf{U} - \mathbf{u})$ . But this expression satisfies

$$\begin{aligned} \mathbf{L}^{M,N}(\mathbf{U} - \mathbf{u}) &= \mathbf{L}^{M,N}(\mathbf{U}) - \mathbf{L}^{M,N}(\mathbf{u}) = \\ \mathbf{f} - \mathbf{L}^{M,N}(\mathbf{u}) &= \mathbf{L}(\mathbf{u}) - \mathbf{L}^{M,N}(\mathbf{u}) = (\mathbf{L} - \mathbf{L}^{M,N})\mathbf{u}. \end{aligned}$$

It follows that

$$\mathbf{L}^{M,N}(\mathbf{U} - \mathbf{u}) = \left(\frac{\partial}{\partial t} - D_t^-\right)\mathbf{u} - E\left(\frac{\partial^2}{\partial x^2} - \delta_x^2\right)\mathbf{u}.$$

Let  $\mathbf{V}, \mathbf{W}^L, \mathbf{W}^R$  be the discrete analogues of  $\mathbf{v}, \mathbf{w}^L, \mathbf{w}^R$  respectively. Then, similarly,

$$\mathbf{L}^{M,N}(\mathbf{V} - \mathbf{v}) = \left(\frac{\partial}{\partial t} - D_t^-\right)\mathbf{v} - E\left(\frac{\partial^2}{\partial x^2} - \delta_x^2\right)\mathbf{v},$$

$$\mathbf{L}^{M,N}(\mathbf{W}^L - \mathbf{w}^L) = \left(\frac{\partial}{\partial t} - D_t^-\right)\mathbf{w}^L - E\left(\frac{\partial^2}{\partial x^2} - \delta_x^2\right)\mathbf{w}^L,$$

$$\mathbf{L}^{M,N}(\mathbf{W}^R - \mathbf{w}^R) = \left(\frac{\partial}{\partial t} - D_t^-\right)\mathbf{w}^R - E\left(\frac{\partial^2}{\partial x^2} - \delta_x^2\right)\mathbf{w}^R,$$

and so, for each  $i = 1, \dots, n$ ,

$$|(\mathbf{L}^{M,N}(\mathbf{V} - \mathbf{v}))_i| \leq \left|\left(\frac{\partial}{\partial t} - D_t^-\right)v_i\right| + |\varepsilon_i\left(\frac{\partial^2}{\partial x^2} - \delta_x^2\right)v_i|, \quad (48)$$

$$|(\mathbf{L}^{M,N}(\mathbf{W}^L - \mathbf{w}^L))_i| \leq \left|\left(\frac{\partial}{\partial t} - D_t^-\right)w_i^L\right| + |\varepsilon_i\left(\frac{\partial^2}{\partial x^2} - \delta_x^2\right)w_i^L|, \quad (49)$$

$$|(\mathbf{L}^{M,N}(\mathbf{W}^R - \mathbf{w}^R))_i| \leq \left|\left(\frac{\partial}{\partial t} - D_t^-\right)w_i^R\right| + |\varepsilon_i\left(\frac{\partial^2}{\partial x^2} - \delta_x^2\right)w_i^R|. \quad (50)$$

Thus, the smooth and singular components of the local truncation error can be treated separately. Note that, for any smooth function  $\psi$ , the following distinct estimates of the local truncation error hold:

for each  $(x_j, t_k) \in \Omega^{L,M}$

$$\left|\left(\frac{\partial}{\partial t} - D_t^-\right)\psi(x_j, t_k)\right| \leq C(t_k - t_{k-1}) \max_{s \in [t_{k-1}, t_k]} \left|\frac{\partial^2 \psi}{\partial t^2}(x_j, s)\right|, \quad (51)$$

$$|(\frac{\partial^2}{\partial x^2} - \delta_x^2)\psi(x_j, t_k)| \leq C \max_{s \in I_j} |\frac{\partial^2 \psi}{\partial x^2}(s, t_k)|, \quad (52)$$

and

$$|(\frac{\partial^2}{\partial x^2} - \delta_x^2)\psi(x_j, t_k)| \leq C(x_{j+1} - x_{j-1}) \max_{s \in I_j} |\frac{\partial^3 \psi}{\partial x^3}(s, t_k)|. \quad (53)$$

Assuming, furthermore, that  $x_j \notin J$ , then

$$|(\frac{\partial^2}{\partial x^2} - \delta_x^2)\psi(x_j, t_k)| \leq C(x_{j+1} - x_{j-1})^2 \max_{s \in I_j} |\frac{\partial^4 \psi}{\partial x^4}(s, t_k)|. \quad (54)$$

Here  $I_j = [x_{j-1}, x_{j+1}]$ .

## 7 Error estimate

The proof of the error estimate is broken into two parts. In the first a theorem concerning the smooth part of the error is proved. Then the singular part of the error is considered. A barrier function is now constructed, which is used in both parts of the proof.

For each  $r \in R$ , introduce a piecewise linear polynomial  $\theta_r$  on  $\overline{\Omega}$ , defined by

$$\theta_r(x) = \begin{cases} \frac{x}{\sigma_r}, & 0 \leq x \leq \sigma_r. \\ 1, & \sigma_r < x < 1 - \sigma_r. \\ \frac{1-x}{\sigma_r}, & 1 - \sigma_r \leq x \leq 1. \end{cases}$$

It is not hard to verify that

$$L^{M,N}(\theta_r(x_j)\mathbf{e})_i \geq \begin{cases} \alpha\theta_r(x_j), & \text{if } x_j \notin J \\ \alpha + \frac{2\varepsilon_i}{\sigma_r(h_r + h_{r+1})}, & \text{if } x_j = \sigma_r \in J. \end{cases} \quad (55)$$

On the Shishkin mesh  $\Omega^{M,N}$  define the barrier function  $\Phi$  by

$$\Phi(x_j, t_k) = C[M^{-1} + N^{-2} + N^{-2}(\ln N)^3 \sum_{r \in R} \theta_r(x_j)]\mathbf{e}, \quad (56)$$

where  $C$  is any sufficiently large constant.

Then, on  $\Omega^{M,N}$ ,  $\Phi$  satisfies

$$0 \leq \Phi_i(x_j, t_k) \leq C(M^{-1} + N^{-2}(\ln N)^3), \quad 1 \leq i \leq n. \quad (57)$$

Also, for  $x_j \notin J$ ,

$$(L^{M,N}\Phi)_i(x_j, t_k) \geq C(M^{-1} + N^{-2}(\ln N)^3) \quad (58)$$

and, for  $x_j \in J$ , using (41), (55),

$$(L^{M,N}\Phi(x_j, t_k))_i \geq \begin{cases} C(M^{-1} + N^{-2} + \frac{\varepsilon_i}{\sqrt{\varepsilon_r \varepsilon_{r+1}}} N^{-1} \ln N), & \text{if } D^+ h_r > 0, \\ C(M^{-1} + N^{-2} + \frac{\varepsilon_i}{\varepsilon_r} N^{-1} \ln N), & \text{if } D^+ h_r < 0. \end{cases} \quad (59)$$

The following theorem gives the required error estimate for the smooth component.

**Theorem 1.** *Let  $A(x, t)$  satisfy (2) and (3). Let  $\mathbf{v}$  denote the smooth component of the exact solution from (1) and  $\mathbf{V}$  the smooth component of the discrete solution from (47). Then*

$$\|\mathbf{V} - \mathbf{v}\| \leq C(M^{-1} + N^{-2}(\ln N)^3). \quad (60)$$

*Proof.* It suffices to show that

$$\frac{|(L^{M,N}(\mathbf{V} - \mathbf{v}))_i(x_j, t_k)|}{|(L^{M,N}\Phi)_i(x_j, t_k)|} \leq C, \quad (61)$$

for each  $i = 1, \dots, n$ , because an application of the Comparison Principle then yields the required result.

For each mesh point  $x_j$  either  $x_j \notin J$  or  $x_j \in J$ .

Suppose first that  $x_j \notin J$ . Then, from (58),

$$(L^{M,N}\Phi(x_j, t_k))_i \geq C(M^{-1} + N^{-2}) \quad (62)$$

and from (51), (54) and Lemma 4

$$\begin{aligned} |(L^{M,N}(\mathbf{V} - \mathbf{v}))_i(x_j, t_k)| &\leq C(t_k - t_{k-1} + (x_{j+1} - x_{j-1})^2) \\ &\leq C(M^{-1} + (h_j + h_{j+1})^2) \\ &\leq C(M^{-1} + N^{-2}). \end{aligned} \quad (63)$$

Then (61) follows from (62) and (63) as required.

On the other hand, when  $x_j \in J$ , by (51), (53) and Lemma 6

$$|(L^{M,N}(\mathbf{V} - \mathbf{v}))_i(x_j, t_k)| \leq C[M^{-1} + \varepsilon_i(h_r + h_{r+1})(1 + \sum_{q=i}^n \frac{B_q(\sigma_r - h_r)}{\sqrt{\varepsilon_q}})]. \quad (64)$$

The cases  $i \geq r$  and  $i < r$  are treated separately.

Suppose first that  $i \geq r$ , then it is not hard to see that

$$\begin{aligned} |(L^{M,N}(\mathbf{V} - \mathbf{v}))_i(x_j, t_k)| &\leq C[M^{-1} + \varepsilon_i(h_r + h_{r+1})(1 + \frac{1}{\sqrt{\varepsilon_i}})] \\ &\leq C[M^{-1} + (h_r + h_{r+1})\sqrt{\varepsilon_i}]. \end{aligned} \quad (65)$$

Combining (59) and (65), (61) follows using (41) and the ordering of the  $\varepsilon_i$ . On the other hand, if  $i < r$  then  $\varepsilon_i \leq \varepsilon_{r-1} < \varepsilon_r$ . Also, either  $d_r > 0$  or  $d_r = 0$ .

First, suppose that  $d_r > 0$ . Then, by Lemma 8,

$$\sigma_r - h_r \geq x_{q,r}^{(\frac{1}{2})}; \text{ for } i \leq q \leq r-1$$

and so, by Lemma 5

$$\sum_{q=i}^{r-1} \frac{B_q(\sigma_r - h_r)}{\sqrt{\varepsilon_q}} \leq C \frac{B_r(\sigma_r - h_r)}{\sqrt{\varepsilon_r}}.$$

Combining this with (64) gives

$$|(L^{M,N}(\mathbf{V} - \mathbf{v}))_i(x_j, t_k)| \leq C[M^{-1} + \frac{\varepsilon_i}{\sqrt{\varepsilon_r}}(h_r + h_{r+1})]. \quad (66)$$

Combining (59) and (66), (61) follows using (41) and the ordering of the  $\varepsilon_i$ . Secondly, suppose that  $d_r = 0$ . Then  $d_{r-1} > 0$  and  $D^+ h_r < 0$ . Then, by Lemma 8 with  $r-1$  instead of  $r$ ,

$$\sigma_r - h_r \geq \sigma_{r-1} > \sigma_{r-1} - h_{r-1} \geq x_{q,r-1}^{(\frac{1}{2})} \text{ for } i \leq q \leq r-2$$

and so, by Lemma 5

$$\sum_{q=i}^{r-2} \frac{B_q(\sigma_r - h_r)}{\sqrt{\varepsilon_q}} \leq C \frac{B_{r-1}(\sigma_r - h_r)}{\sqrt{\varepsilon_{r-1}}} \leq C \frac{B_{r-1}(\sigma_{r-1})}{\sqrt{\varepsilon_{r-1}}} = C \frac{N^{-2}}{\sqrt{\varepsilon_{r-1}}}.$$

Combining this with (64) and (41) gives

$$\begin{aligned} & |(L^{M,N}(\mathbf{V} - \mathbf{v}))_i(x_j, t_k)| \\ & \leq C[M^{-1} + \varepsilon_i(h_r + h_{r+1})\left(\frac{N^{-2}}{\sqrt{\varepsilon_{r-1}}} + \frac{1}{\sqrt{\varepsilon_r}}\right)] \\ & \leq C[M^{-1} + \sqrt{\varepsilon_i \varepsilon_r} N^{-3} \ln N + \varepsilon_i N^{-1} \ln N]. \end{aligned} \quad (67)$$

Combining (59) and (67), (61) follows using the ordering of the  $\varepsilon_i$  and noting that in this case the middle term in the denominator is used to bound the middle term in the numerator.

■

Before the singular part of the error is estimated the following lemmas are established.

**Lemma 12.** *Assume that  $x_j \notin J$ . Let  $A(x, t)$  satisfy (2) and (3). Then, on  $\Omega^{M,N}$ , for each  $1 \leq i \leq n$ , the following estimates hold*

$$|(L^{M,N}(\mathbf{W}^{\mathbf{L}} - \mathbf{w}^{\mathbf{L}}))_i(x_j, t_k)| \leq C(M^{-1} + \frac{(x_{j+1} - x_{j-1})^2}{\varepsilon_1}). \quad (68)$$

An analogous result holds for the  $w_i^R$ .

*Proof.* Since  $x_j \notin J$ , from (54) and Lemma 7, it follows that

$$\begin{aligned} |(L^{M,N}(\mathbf{W}^L - \mathbf{w}^L))_i(x_j, t_k)| &= |((\frac{\partial}{\partial t} - D_t^-) - E(\frac{\partial^2}{\partial x^2} - \delta_x^2))\mathbf{w}_i^L(x_j, t_k)| \\ &\leq C(M^{-1} + (x_{j+1} - x_{j-1})^2) \max_{s \in I_j} \sum_{q=1}^n \frac{B_q^L(s)}{\varepsilon_q} \\ &\leq C(M^{-1} + \frac{(x_{j+1} - x_{j-1})^2}{\varepsilon_1}) \end{aligned}$$

as required. ■

The following decompositions are introduced

$$w_i^L = \sum_{m=1}^{r+1} w_{i,m},$$

where the components are defined by

$$w_{i,r+1} = \begin{cases} p_i^{(s)} & \text{on } [0, x_{r,r+1}^{(s)}) \\ w_i^L & \text{otherwise} \end{cases}$$

and for each  $m, r \geq m \geq 2$ ,

$$w_{i,m} = \begin{cases} p_i^{(s)} & \text{on } [0, x_{m-1,m}^{(s)}) \\ w_i^L - \sum_{q=m+1}^{r+1} w_{i,q} & \text{otherwise} \end{cases}$$

and

$$w_{i,1} = w_i^L - \sum_{q=2}^{r+1} w_{i,q} \quad \text{on } [0, 1].$$

Here the polynomials  $p_i^{(s)}$ , for  $s = 3/2$  and  $s = 1$ , are defined by

$$p_i^{(3/2)}(x, t) = \sum_{q=0}^3 \frac{\partial^q w_i^{(L)}}{\partial x^q}(x_{r,r+1}^{(3/2)}, t) \frac{(x - x_{r,r+1}^{(3/2)})^q}{q!}$$

and

$$p_i^{(1)}(x, t) = \sum_{q=0}^4 \frac{\partial^q w_i^{(L)}}{\partial x^q}(x_{r,r+1}^{(1)}, t) \frac{(x - x_{r,r+1}^{(1)})^q}{q!}.$$

**Lemma 13.** Assume that  $d_r > 0$ . Let  $A(x, t)$  satisfy (2) and (3). Then, for each  $1 \leq i \leq n$ , there exists a decomposition

$$w_i^L = \sum_{q=1}^{r+1} w_{i,q},$$

for which the following estimates hold for each  $q$  and  $r$ ,  $1 \leq q \leq r$ ,

$$\begin{aligned} \left| \frac{\partial^2 w_{i,q}}{\partial x^2}(x_j, t_k) \right| &\leq C \min\left\{ \frac{1}{\varepsilon_q}, \frac{1}{\varepsilon_i} \right\} B_q^L(x_j), \\ \left| \frac{\partial^3 w_{i,q}}{\partial x^3}(x_j, t_k) \right| &\leq C \min\left\{ \frac{1}{\varepsilon_q^{3/2}}, \frac{1}{\varepsilon_i \sqrt{\varepsilon_q}} \right\} B_q^L(x_j), \\ \left| \frac{\partial^3 w_{i,r+1}}{\partial x^3}(x_j, t_k) \right| &\leq C \min\left\{ \sum_{q=r+1}^n \frac{B_q^L(x_j)}{\varepsilon_q^{3/2}}, \sum_{q=r+1}^n \frac{B_q^L(x_j)}{\varepsilon_i \sqrt{\varepsilon_q}} \right\}, \\ \left| \frac{\partial^4 w_{i,q}}{\partial x^4}(x_j, t_k) \right| &\leq C \frac{B_q^L(x_j)}{\varepsilon_i \varepsilon_q}, \\ \left| \frac{\partial^4 w_{i,r+1}}{\partial x^4}(x_j, t_k) \right| &\leq C \sum_{q=r+1}^n \frac{B_q^L(x_j)}{\varepsilon_i \varepsilon_q}. \end{aligned}$$

Analogous results hold for the  $w_i^R$  and their derivatives.

*Proof.* First consider the decomposition corresponding  $s = 3/2$ .

From the above definitions it follows that, for each  $m$ ,  $1 \leq m \leq r$ ,  $w_{i,m} = 0$  on  $[x_{m,m+1}^{(3/2)}, 1]$ .

To establish the bounds on the third derivatives it is seen that: for  $x \in [x_{r,r+1}^{(3/2)}, 1]$ , Lemma 7 and  $x \geq x_{r,r+1}^{(3/2)}$  imply that

$$\left| \frac{\partial^3 w_{i,r+1}}{\partial x^3}(x, t) \right| = \left| \frac{\partial^3 w_i^L}{\partial x^3}(x, t) \right| \leq C \sum_{q=1}^n \frac{B_q^L(x)}{\varepsilon_q^{3/2}} \leq C \sum_{q=r+1}^n \frac{B_q^L(x)}{\varepsilon_q^{3/2}};$$

for  $x \in [0, x_{r,r+1}^{(3/2)}]$ , Lemma 7 and  $x \leq x_{r,r+1}^{(3/2)}$  imply that

$$\begin{aligned} \left| \frac{\partial^3 w_{i,r+1}}{\partial x^3}(x, t) \right| &= \left| \frac{\partial^3 w_i^L}{\partial x^3}(x_{r,r+1}^{(3/2)}, t) \right| \\ &\leq \sum_{q=1}^n \frac{B_q^L(x_{r,r+1}^{(3/2)})}{\varepsilon_q^{3/2}} \leq \sum_{q=r+1}^n \frac{B_q^L(x_{r,r+1}^{(3/2)})}{\varepsilon_q^{3/2}} \leq \sum_{q=r+1}^n \frac{B_q^L(x)}{\varepsilon_q^{3/2}}; \end{aligned}$$

and for each  $m = r, \dots, 2$ , it follows that

for  $x \in [x_{m,m+1}^{(3/2)}, 1]$ ,  $w_{i,m}^{(3)} = 0$ ;

for  $x \in [x_{m-1,m}^{(3/2)}, x_{m,m+1}^{(3/2)}]$ , Lemma 7 implies that



$$\begin{aligned}
\left| \frac{\partial^3 w_{i,m}}{\partial x^3}(x, t) \right| &\leq \left| \frac{\partial^3 w_i^L}{\partial x^3}(x, t) \right| + \sum_{q=m+1}^{r+1} \left| \frac{\partial^3 w_{i,q}}{\partial x^3}(x, t) \right| \\
&\leq C \sum_{q=1}^n \frac{B_q^L(x)}{\varepsilon_q^{3/2}} \leq C \frac{B_m^L(x)}{\varepsilon_m^{3/2}},
\end{aligned}$$

for  $x \in [0, x_{m-1,m}^{(3/2)}]$ , Lemma 7 and  $x \leq x_{m-1,m}^{(3/2)}$  imply that

$$\begin{aligned}
\left| \frac{\partial^3 w_{i,m}}{\partial x^3}(x, t) \right| &= \left| \frac{\partial^3 w_i^L}{\partial x^3}(x_{m-1,m}^{(3/2)}, t) \right| \\
&\leq C \sum_{q=1}^n \frac{B_q^L(x_{m-1,m}^{(3/2)})}{\varepsilon_q^{3/2}} \leq C \frac{B_m^L(x_{m-1,m}^{(3/2)})}{\varepsilon_m^{3/2}} \leq C \frac{B_m^L(x)}{\varepsilon_m^{3/2}},
\end{aligned}$$

for  $x \in [x_{1,2}^{(3/2)}, 1]$ ,  $\frac{\partial^3 w_{i,1}}{\partial x^3} = 0$ ;

for  $x \in [0, x_{1,2}^{(3/2)}]$ , Lemma 7 implies that

$$\left| \frac{\partial^3 w_{i,1}}{\partial x^3}(x, t) \right| \leq \left| \frac{\partial^3 w_i^L}{\partial x^3}(x, t) \right| + \sum_{q=2}^{r+1} \left| \frac{\partial^3 w_{i,q}}{\partial x^3}(x, t) \right| \leq C \sum_{q=1}^n \frac{B_q^L(x)}{\varepsilon_q^{3/2}} \leq C \frac{B_1^L(x)}{\varepsilon_1^{3/2}}.$$

For the bounds on the second derivatives note that, for each  $m$ ,  $1 \leq m \leq r$

$$\begin{aligned}
&: \text{ for } x \in [x_{m,m+1}^{(3/2)}, 1], \quad \frac{\partial^2 w_{i,m}}{\partial x^2} = 0; \\
& \text{ for } x \in [0, x_{m,m+1}^{(3/2)}], \quad \int_x^{x_{m,m+1}^{(3/2)}} \frac{\partial^3 w_{i,m}}{\partial x^3}(s, t) ds = \\
& \frac{\partial^2 w_{i,m}}{\partial x^2}(x_{m-1,m}^{(3/2)}, t) - \frac{\partial^2 w_{i,m}}{\partial x^2}(x, t) = -\frac{\partial^2 w_{i,m}}{\partial x^2}(x, t)
\end{aligned}$$

and so

$$\left| \frac{\partial^2 w_{i,m}}{\partial x^2}(x, t) \right| \leq \int_x^{x_{m,m+1}^{(3/2)}} \left| \frac{\partial^3 w_{i,m}}{\partial x^3}(s, t) \right| ds \leq \frac{C}{\varepsilon_m^{3/2}} \int_x^{x_{m,m+1}^{(3/2)}} B_m^L(s) ds \leq C \frac{B_m^L(x)}{\varepsilon_m}.$$

This completes the proof of the estimates for  $s = 3/2$ .

For the estimates in the case  $s = 1$  consider the decomposition

$$w_i^L = \sum_{m=1}^{r+1} w_{i,m}.$$

From the above definitions it follows that, for each  $m$ ,  $1 \leq m \leq r$ ,  $w_{i,m} = 0$  on  $[x_{m,m+1}^{(1)}, 1]$ .

To establish the bounds on the fourth derivatives it is seen that:

for  $x \in [x_{r,r+1}^{(1)}, 1]$ , Lemma 7 and  $x \geq x_{r,r+1}^{(1)}$  imply that

$$|\varepsilon_i \frac{\partial^4 w_{i,r+1}}{\partial x^4}(x, t)| = |\varepsilon_i \frac{\partial^4 w_i^L}{\partial x^4}(x, t)| \leq C \sum_{q=1}^n \frac{B_q^L(x)}{\varepsilon_q} \leq C \sum_{q=r+1}^n \frac{B_q^L(x)}{\varepsilon_q};$$

for  $x \in [0, x_{r,r+1}^{(1)}]$ , Lemma 7 and  $x \leq x_{r,r+1}^{(1)}$  imply that

$$\begin{aligned} |\varepsilon_i \frac{\partial^4 w_{i,r+1}}{\partial x^4}(x, t)| &= |\varepsilon_i \frac{\partial^4 w_i^L}{\partial x^4}(x_{r,r+1}^{(1)}, t)| \\ &\leq \sum_{q=1}^n \frac{B_q^L(x_{r,r+1}^{(1)})}{\varepsilon_q} \leq C \sum_{q=r+1}^n \frac{B_q^L(x_{r,r+1}^{(1)})}{\varepsilon_q} \leq C \sum_{q=r+1}^n \frac{B_q^L(x)}{\varepsilon_q}; \end{aligned}$$

and for each  $m = r, \dots, 2$ , it follows that

for  $x \in [x_{m,m+1}^{(1)}, 1]$ ,  $\frac{\partial^4 w_{i,m}}{\partial x^4} = 0$ ;

for  $x \in [x_{m-1,m}^{(1)}, x_{m,m+1}^{(1)}]$ , Lemma 7 implies that

$$\begin{aligned} |\varepsilon_i \frac{\partial^4 w_{i,m}}{\partial x^4}(x, t)| &\leq |\varepsilon_i \frac{\partial^4 w_i^L}{\partial x^4}(x, t)| + \sum_{q=m+1}^{r+1} |\varepsilon_i \frac{\partial^4 w_{i,q}}{\partial x^4}(x, t)| \\ &\leq C \sum_{q=1}^n \frac{B_q^L(x)}{\varepsilon_q} \leq C \frac{B_m^L(x)}{\varepsilon_m}; \end{aligned}$$

for  $x \in [0, x_{m-1,m}^{(1)}]$ , Lemma 7 and  $x \leq x_{m-1,m}^{(1)}$  imply that

$$\begin{aligned} |\varepsilon_i \frac{\partial^4 w_{i,m}}{\partial x^4}(x, t)| &= |\varepsilon_i \frac{\partial^4 w_i^L}{\partial x^4}(x_{m-1,m}^{(1)}, t)| \\ &\leq C \sum_{q=1}^n \frac{B_q^L(x_{m-1,m}^{(1)})}{\varepsilon_q} \leq C \frac{B_m^L(x_{m-1,m}^{(1)})}{\varepsilon_m} \leq C \frac{B_m^L(x)}{\varepsilon_m}; \end{aligned}$$

for  $x \in [x_{1,2}^{(1)}, 1]$ ,  $\frac{\partial^4 w_{i,1}}{\partial x^4} = 0$ ;

for  $x \in [0, x_{1,2}^{(1)}]$ , Lemma 7 implies that

$$\begin{aligned} |\varepsilon_i \frac{\partial^4 w_{i,1}}{\partial x^4}(x, t)| &\leq |\varepsilon_i \frac{\partial^4 w_i^L}{\partial x^4}(x, t)| + \sum_{q=2}^{r+1} |\varepsilon_i \frac{\partial^4 w_{i,q}}{\partial x^4}(x, t)| \\ &\leq C \sum_{q=1}^n \frac{B_q^L(x)}{\varepsilon_q} \leq C \frac{B_1^L(x)}{\varepsilon_1}. \end{aligned}$$

For the bounds on the second and third derivatives note that, for each  $m$ ,  $1 \leq m \leq r$ :

for  $x \in [x_{m,m+1}^{(1)}, 1]$ ,  $\frac{\partial^2 w_{i,m}}{\partial x^2} = 0 = \frac{\partial^3 w_{i,m}}{\partial x^3}$ ;

$$\begin{aligned}
& \text{for } x \in [0, x_{m,m+1}^{(1)}], \quad \int_x^{x_{m,m+1}^{(1)}} \varepsilon_i \frac{\partial^4 w_{i,m}}{\partial x^4}(s, t) ds \\
&= \varepsilon_i \frac{\partial^3 w_{i,m}}{\partial x^3}(x_{m,m+1}^{(1)}, t) - \varepsilon_i \frac{\partial^3 w_{i,m}}{\partial x^3}(x, t) = -\varepsilon_i \frac{\partial^3 w_{i,m}}{\partial x^3}(x, t)
\end{aligned}$$

and so

$$|\varepsilon_i \frac{\partial^3 w_{i,m}}{\partial x^3}(x, t)| \leq \int_x^{x_{m,m+1}^{(1)}} |\varepsilon_i \frac{\partial^4 w_{i,m}}{\partial x^4}(s, t)| ds \leq \frac{C}{\varepsilon_m} \int_x^{x_{m,m+1}^{(1)}} B_m^L(s) ds \leq C \frac{B_m^L(x)}{\sqrt{\varepsilon_m}}.$$

In a similar way, it can be shown that

$$|\varepsilon_i \frac{\partial^2 w_{i,m}}{\partial x^2}(x, t)| \leq C B_m^L(x).$$

The proof for the  $w_i^R$  and their derivatives is similar. ■

**Lemma 14.** Assume that  $d_r > 0$ . Let  $A(x)$  satisfy (2) and (3). Then, for each  $i$ ,  $1 \leq i \leq n$ , and each  $(x_j, t_k) \in \Omega^{M,N}$

$$|(L^{M,N}(\mathbf{W}^L - \mathbf{w}^L)_i(x_j, t_k))| \leq C[M^{-1} + B_r^L(x_{j-1}) + \frac{x_{j+1} - x_{j-1}}{\sqrt{\varepsilon_{r+1}}}] \quad (69)$$

and

$$|(L^{M,N}(\mathbf{W}^L - \mathbf{w}^L)_i(x_j, t_k))| \leq C[M^{-1} + \varepsilon_i \sum_{q=1}^r \frac{B_q^L(x_{j-1})}{\varepsilon_q} + \frac{\varepsilon_i}{\varepsilon_{r+1}} \frac{x_{j+1} - x_{j-1}}{\sqrt{\varepsilon_{r+1}}}] \quad (70)$$

Assuming, furthermore, that  $x_j \notin J$ , then

$$|(L^{M,N}(\mathbf{W}^L - \mathbf{w}^L)_i(x_j, t_k))| \leq C[M^{-1} + B_r^L(x_{j-1}) + \frac{(x_{j+1} - x_{j-1})^2}{\varepsilon_{r+1}}]. \quad (71)$$

Analogous results hold for the  $W^R - w_i^R$  and their derivatives.

*Proof.* Using (49), (51) and the bound in Lemma 7, for any  $(x_j, t_k) \in \Omega^{M,N}$ ,

$$|(L^{M,N}(\mathbf{W}^L - \mathbf{w}^L)_i(x_j, t_k))| \leq C[(t_k - t_{k-1}) + |\varepsilon_i(\delta_x^2 - \frac{\partial^2}{\partial x^2})w_i^L(x_j, t_k)|]. \quad (72)$$

From the decompositions and bounds in Lemma 13, with (52) and (53), it follows from (72) that

$$\begin{aligned}
& |\varepsilon_i(\delta_x^2 - \frac{\partial^2}{\partial x^2})w_i^L(x_j, t_k)| \\
& \leq C[\sum_{q=1}^r |\varepsilon_i(D_x^2 - \frac{\partial^2}{\partial x^2})w_{i,q}(x_j, t_k)| + |\varepsilon_i(D_x^2 - \frac{\partial^2}{\partial x^2})w_{i,r+1}(x_j, t_k)|] \\
& \leq C[\sum_{q=1}^r \max_{s \in I_j} |\varepsilon_i w_{i,q}^{(2)}(s, t_k)| + (x_{j+1} - x_{j-1}) \max_{s \in I_j} |\varepsilon_i w_{i,r+1}^{(3)}(s, t_k)|] \\
& \leq C[\sum_{q=1}^r \min\{\frac{\varepsilon_i}{\varepsilon_q}, 1\} B_q^L(x_{j-1}) + (x_{j+1} - x_{j-1}) \min\{\frac{\varepsilon_i}{\varepsilon_{r+1}}, 1\} \frac{B_{r+1}^L(x_{j-1})}{\sqrt{\varepsilon_{r+1}}}] .
\end{aligned} \tag{73}$$

Substituting 1 for each of the min expressions gives (69) and (70) is obtained by substituting the appropriate ratio  $\frac{\varepsilon_i}{\varepsilon_q}$  in each such expression.

In the remaining case when  $x_j \notin J$ , (54) can be used instead of (53), and it follows by a similar argument to the above that

$$|\varepsilon_i(\delta_x^2 - \frac{\partial^2}{\partial x^2})w_i^L(x_j, t_k)| \leq C[\sum_{q=1}^r \min\{\frac{\varepsilon_i}{\varepsilon_q}, 1\} B_q^L(x_{j-1}) + \frac{(x_{j+1} - x_{j-1})^2}{\varepsilon_{r+1}}]. \tag{74}$$

Substituting 1 for the min expression, as before, gives (71).

The proof for the  $w_i^R$  and their derivatives is similar. ■

**Lemma 15.** *Let  $A(x, t)$  satisfy (2) and (3). Then, on  $\Omega^{M,N}$ , for each  $1 \leq i \leq n$ , the following estimates hold*

$$|(L^{M,N}(\mathbf{W}^L - \mathbf{w}^L))_i(x_j, t_k)| \leq C(M^{-1} + B_n^L(x_{j-1})). \tag{75}$$

An analogous result holds for the  $w_i^R$ .

*Proof.* From (52) and Lemma 7, for each  $i = 1, \dots, n$ , it follows that on  $\Omega^{M,N}$ ,

$$\begin{aligned}
|(L^{M,N}(\mathbf{W}^L - \mathbf{w}^L))_i(x_j, t_k)| &= |((\frac{\partial}{\partial t} - D_t^-) - E(\frac{\partial^2}{\partial x^2} - \delta_x^2))\mathbf{w}_i^L(x_j, t_k)| \\
&\leq C(M^{-1} + \varepsilon_i \sum_{q=i}^n \frac{B_q^L(x_{j-1})}{\varepsilon_q}) \\
&\leq C(M^{-1} + B_n^L(x_{j-1})).
\end{aligned}$$

The proof for the  $w_i^R$  and their derivatives is similar. ■

The following theorem provides the error estimate for the singular component.

**Theorem 2.** *Let  $A(x, t)$  satisfy (2) and (3). Let  $\mathbf{w}$  denote the singular component of the exact solution from (1) and  $\mathbf{W}$  the singular component of the discrete solution from (47). Then*

$$\|\mathbf{W} - \mathbf{w}\| \leq C(M^{-1} + N^{-2}(\ln N)^3). \tag{76}$$

*Proof.* Since  $\mathbf{w} = \mathbf{w}^L + \mathbf{w}^R$ , it suffices to prove the result for  $\mathbf{w}^L$  and  $\mathbf{w}^R$  separately. Here it is proved for  $\mathbf{w}^L$  by an application of Lemma 11. A similar proof holds for  $\mathbf{w}^R$ .

The proof is in two parts:  $x_j$  is such that either  $x_j \notin J$  or  $x_j = \sigma_r \in J$ .

First assume that  $x_j \notin J$ . Each open subinterval  $(\sigma_k, \sigma_{k+1})$  is treated separately.

First, consider  $x_j \in (0, \sigma_1)$ . Then, on each mesh  $M$ ,  $x_{j+1} - x_{j-1} \leq CN^{-1}\sigma_1$  and the result follows from (40) and Lemma 12.

Secondly, consider  $x_j \in (\sigma_1, \sigma_2)$ , then  $\sigma_1 \leq x_{j-1}$  and  $x_{j+1} - x_{j-1} \leq CN^{-1}\sigma_2$ . The  $2^n$  possible meshes are divided into subclasses of two types. On the meshes  $\overline{\Omega}^{M,N}$  with  $b_1 = 0$  the result follows from (40), (42) and Lemma 12. On the meshes  $\overline{\Omega}^{M,N}$  with  $b_1 = 1$  the result follows from (40), (43) and Lemma 13.

Thirdly, in the general case  $x_j \in (\sigma_m, \sigma_{m+1})$  for  $2 \leq m \leq n-1$ , it follows that  $\sigma_m \leq x_{j-1}$  and  $x_{j+1} - x_{j-1} \leq CN^{-1}\sigma_{m+1}$ . Then  $\overline{\Omega}^{M,N}$  is divided into subclasses of three types:  $\overline{\Omega}_0^{M,N} = \{\overline{\Omega}^{M,N} : b_1 = \dots = b_m = 0\}$ ,  $\overline{\Omega}_r^{M,N} = \{\overline{\Omega}^{M,N} : b_r = 1, b_{r+1} = \dots = b_m = 0 \text{ for some } 1 \leq r \leq m-1\}$  and  $\overline{\Omega}_m^{M,N} = \{\overline{\Omega}^{M,N} : b_m = 1\}$ . On  $\overline{\Omega}_0^{M,N}$  the result follows from (40), (42) and Lemma 12; on  $\overline{\Omega}_r^{M,N}$  from (40), (42), (43) and Lemma 13; on  $\overline{\Omega}_m^{M,N}$  from (40), (43) and Lemma 13.

Finally, for  $x_j \in (\sigma_n, 1)$ ,  $\sigma_n \leq x_{j-1}$  and  $x_{j+1} - x_{j-1} \leq CN^{-1}$ . Then  $\overline{\Omega}^{M,N}$  is divided into subclasses of three types:  $\overline{\Omega}_0^{M,N} = \{\overline{\Omega}^{M,N} : b_1 = \dots = b_n = 0\}$ ,  $\overline{\Omega}_r^{M,N} = \{\overline{\Omega}^{M,N} : b_r = 1, b_{r+1} = \dots = b_n = 0 \text{ for some } 1 \leq r \leq n-1\}$  and  $\overline{\Omega}_n^{M,N} = \{\overline{\Omega}^{M,N} : b_n = 1\}$ . On  $\overline{\Omega}_0^{M,N}$  the result follows from (40), (42) and Lemma 12; on  $\overline{\Omega}_r^{M,N}$  from (40), (42), (43) and Lemma 13; on  $\overline{\Omega}_n^{M,N}$  from (43) and Lemma 15.

Now assume that  $x_j \in J$ . Then  $x_j = \sigma_r$ , some  $r$ . It suffices to show that

$$\frac{|(L^{M,N}(\mathbf{W}^L - \mathbf{w}^L))_i(x_j, t_k)|}{|(L^{M,N}\Phi)_i(x_j, t_k)|} \leq C, \quad (77)$$

for each  $i = 1, \dots, n$ , because an application of the Comparison Principle then yields the required result.

The bounds on the denominator are given in (59). To bound the numerator note that either  $d_r > 0$  or  $d_r = 0$ .

Suppose first that  $d_r > 0$ . Then the cases  $i > r$  and  $i \leq r$  are treated separately.

If  $i > r$ , then, by (69) in Lemma 14,

$$|(L^{M,N}(\mathbf{W}^L - \mathbf{w}^L))_i(x_j, t_k)| \leq C[M^{-1} + B_r^L(x_{j-1}) + \frac{x_{j+1} - x_{j-1}}{\sqrt{\varepsilon_{r+1}}}] \quad (78)$$

Since  $d_r > 0$ , by Lemma 8,  $B_r^L(x_{j-1}) = B_r^L(\sigma_r - h_r) \leq CB_r^L(\sigma_r) = CN^{-2}$ , and so

$$|(L^{M,N}(\mathbf{W}^L - \mathbf{w}^L))_i(x_j, t_k)| \leq C[M^{-1} + N^{-2} + \frac{h_r + h_{r+1}}{\sqrt{\varepsilon_{r+1}}}]. \quad (79)$$

Using (41) and the ordering of the  $\varepsilon_i$ , these bounds on the numerator and denominator lead to (77).

If  $i \leq r$ , then, by (70) in Lemma 14,

$$|(L^{M,N}(\mathbf{W}^L - \mathbf{w}^L))_i(x_j, t_k)| \leq C[M^{-1} + \varepsilon_i \sum_{q=1}^r \frac{B_q^L(x_{j-1})}{\varepsilon_q} + \frac{\varepsilon_i}{\varepsilon_{r+1}} \frac{x_{j+1} - x_{j-1}}{\sqrt{\varepsilon_{r+1}}}] \quad (80)$$

Since  $d_r > 0$ , by Lemma 8,  $x_{j-1} = \sigma_r - h_r \geq x_{q,r}^{(s)}$  for  $1 \leq q \leq r-1$  and

$$\frac{B_q^L(x_{j-1})}{\varepsilon_q} \leq \frac{B_r^L(x_{j-1})}{\varepsilon_r} \leq \frac{B_r^L(\sigma_r)}{\varepsilon_r} = C \frac{N^{-2}}{\varepsilon_r}.$$

Thus

$$|(L^{M,N}(\mathbf{W}^L - \mathbf{w}^L))_i(x_j, t_k)| \leq C[M^{-1} + \frac{\varepsilon_i}{\varepsilon_r} N^{-2} + \frac{\varepsilon_i}{\varepsilon_{r+1}} \frac{x_{j+1} - x_{j-1}}{\sqrt{\varepsilon_{r+1}}}] \quad (81)$$

Using (41) and the ordering of the  $\varepsilon_i$ , these bounds on the numerator and denominator lead to (77).

Now suppose that  $d_r = 0$ . Then  $d_{r-1} > 0$  and  $D^+ h_r < 0$ , because otherwise  $x_j \notin J$ . The cases  $i \geq r$  and  $i < r$  are now treated separately.

If  $i \geq r$ , then, by (69) in Lemma 14 with  $r$  replaced by  $r-1$ ,

$$|(L^{M,N}(\mathbf{W}^L - \mathbf{w}^L))_i(x_j, t_k)| \leq C[M^{-1} + B_{r-1}^L(x_{j-1}) + \frac{x_{j+1} - x_{j-1}}{\sqrt{\varepsilon_r}}] \quad (82)$$

Since  $d_{r-1} > 0$ , by Lemma 8 with  $r$  replaced by  $r-1$ ,

$$B_{r-1}^L(x_{j-1}) = B_{r-1}^L(\sigma_r - h_r) \leq CB_{r-1}^L(\sigma_{r-1}) = CN^{-2},$$

and so

$$|(L^{M,N}(\mathbf{W}^L - \mathbf{w}^L))_i(x_j, t_k)| \leq C[M^{-1} + N^{-2} + \frac{h_r + h_{r+1}}{\sqrt{\varepsilon_r}}]. \quad (83)$$

Using (41) and the ordering of the  $\varepsilon_i$ , these bounds on the numerator and denominator lead to (77).

If  $i < r$ , then by (70) in Lemma 14 with  $r$  replaced by  $r-1$ ,

$$|(L^{M,N}(\mathbf{W}^L - \mathbf{w}^L))_i(x_j, t_k)| \leq C[M^{-1} + \varepsilon_i \sum_{q=1}^{r-1} \frac{B_q^L(x_{j-1})}{\varepsilon_q} + \frac{\varepsilon_i}{\varepsilon_r} \frac{x_{j+1} - x_{j-1}}{\sqrt{\varepsilon_r}}]. \quad (84)$$

Since  $d_{r-1} > 0$ , by Lemma 8 with  $r$  replaced by  $r-1$ ,  $x_{j-1} = \sigma_r - h_r \geq x_{q,r}^{(s)}$  for  $1 \leq q \leq r-1$  and

$$\frac{B_q^L(x_{j-1})}{\varepsilon_q} \leq \frac{B_{r-1}^L(x_{j-1})}{\varepsilon_{r-1}} \leq \frac{B_{r-1}^L(\sigma_{r-1})}{\varepsilon_{r-1}} = C \frac{N^{-2}}{\varepsilon_{r-1}}.$$

Thus

$$|(L^{M,N}(\mathbf{W}^L - \mathbf{w}^L))_i(x_j, t_k)| \leq C[M^{-1} + \frac{\varepsilon_i}{\varepsilon_{r-1}} N^{-2} + \frac{\varepsilon_i}{\varepsilon_r} \frac{x_{j+1} - x_{j-1}}{\sqrt{\varepsilon_r}}]. \quad (85)$$

Using (41) and the ordering of the  $\varepsilon_i$ , these bounds on the numerator and denominator lead to (77). This completes the proof. ■

The following theorem gives the required first order in time and essentially second order in space parameter-uniform error estimate.

**Theorem 3.** *Let  $A(x, t)$  satisfy (2) and (3). Let  $\mathbf{u}$  denote the exact solution of (1) and  $\mathbf{U}$  the discrete solution of (47). Then*

$$\|\mathbf{U} - \mathbf{u}\| \leq C N^{-2} (\ln N)^3. \quad (86)$$

*Proof.* An application of the triangle inequality and the results of Theorems 1 and 2 leads immediately to the required result. ■

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